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On Differential Equations for Orthogonal Polynomials

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1. As well known the so called classical orthogonal polynomials satisfy a recursion formula of type

$$y_n(x)=(x-a_n)y_{n-1}(x)-b_n y_{n-2}(x) \tag{1}$$

as well as a linear differential equation

$$p_{n0}(x)y_n''(x)+p_{n1}(x)y_n'(x)+p_{n2}(x)y_n(x)=0 \tag{2}$$

with polynomial coefficients. The difference equation (1) is characteristic for orthogonal polynomials at all if a_n is real and b_n is positive: each sequence $\{y_n\}$ of orthogonal polynomials satisfy such a formula and, conversely, if the sequence $\{y_n\}$ is defined by (1), the polynomials form an orthogonal system. The question of whether nonclassical orthogonal polynomials can be characterized by a linear differential equation with polynomial coefficients has been dealt with repeatedly (Shohat [7], Hahn [1], Krall [4, 5], Varma [8]) but no general result has been published so far.

I show in the present paper that the minimal order of a linear differential equation for orthogonal polynomials can only take the values two and four, and that, in the latter case, the solutions can be constructed by means of the solutions of second order differential equations. I further give some necessary conditions for the parameters occuring in the case of order two.

2. The argument is essentially based upon the following theorem (Hahn [2]): Suppose we are given two sequences $\{a_\alpha\}$, $\{b_\alpha\}$, $\alpha=\alpha_0, \alpha_0+1, \alpha_0+2, \cdots$, of real or complex numbers and let a sequence $\{y_\alpha(x)\}$ of functions be defined by the recursion formula

$$y_\alpha(x)=(x-a_\alpha)y_{\alpha-1}(x)-b_\alpha y_{\alpha-2}(x). \tag{3}$$

Suppose further that a linear differential equation of order k

$$L_\alpha(y):=p_{\alpha0}y^{(k)}+p_{\alpha1}y^{(k-1)}+\cdots+p_{\alpha k}y=0 \tag{4}$$

exists for each value of $\alpha \geq \alpha_0$ which is fulfilled by $y_\alpha(x)$. The coefficients $p_{\alpha j}$ are polynomials with degrees uniformly bounded with respect to α, and the highest coefficient of $p_{\alpha 0}$ is one. The function $y_\alpha(x)$ does not satisfy such an equation of lower order than k. Suppose finally that the equations $L_\alpha(y)=0$ and $L_{\alpha-2}(y)=0$ have no solutions in common. Then the following statements hold:

a) The genuine singularities of the equation $L_\alpha(y)$ do not depend upon α. Zeroes of $p_{\alpha 0}(x)$ which depend upon α define nebenpoints (apparent singularities).

b) If the variable x moves along a closed path round a singularity, the linear substitution which transforms the fundamental system is independent of α.

c) It is possible, for each value of α, to choose a fundamental system of (4) such that the suitably numerated functions satisfy the recursion formula (3).

If α is an integer, the functions $y_\alpha(x)=y_n(x)$ may be polynomials but the theorem holds generally for solutions of the difference equation (3).

If, in the following, the expression "fundamental system of (4)" is used, it means the special system

$$w_{\alpha 1},\, w_{\alpha 2},\, \cdots,\, w_{\alpha k} \tag{5}$$

which is characterized by the statement c) of the theorem.

2. The difference equation (3) is of order two. Consequently, two functions $u_\alpha(x)$ and $v_\alpha(x)$ can be chosen from the system (5) such that the remaining $k-2$ functions are linear combinations with coefficients which are independent of α but may depend upon x. We have either

$$w_{\alpha i}=f_i(x)u_\alpha(x) \quad \text{or} \quad =g_i(x)v_\alpha(x), \tag{6}$$

respectively, or we have

$$w_{\alpha j}(x)=f_j(x)u_\alpha(x)+g_j(x)v_\alpha(x), \tag{7}$$

where the right hand terms do not fulfil the equation (4) individually.

For the present we exclude the case (7) which will be considered later on and assume that the fundamental system (5) consists of functions

$$f_1u_\alpha, f_2u_\alpha, \cdots, f_ru_\alpha; \qquad g_1v_\alpha, g_2v_\alpha, \cdots, g_sv_\alpha \tag{8}$$

with $f_1=g_1=1$, $r+s=k$.

3. We move the variable x round a singularity, regarding statement b) of the theorem. The matrix of the substitution can be written in the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

A and D are square of order r and s, respectively, corresponding to the structure of (8). The transformed functions are noted by an asterisk. We have

$$f_i^* u_\alpha^* = (f_i u_\alpha)^*, \qquad i=1, 2, \cdots, r,$$

$$\begin{aligned} f_i^*[u_\alpha(a_{11}f_1+\cdots+a_{1r}f_r)+v_\alpha(b_{11}g_1+\cdots+b_{1r}g_s)] \\ = u_\alpha(a_{i1}f_1+\cdots+a_{ir}f_r)+v_\alpha(b_{i1}g_1+\cdots+b_{is}g_s). \end{aligned} \tag{9}$$

The function f_i^* can be written as a quotient, and because f_i^* is independent of α, we conclude that numerator and denominator have a common factor involving u_α and v_α. We obtain

$$a_{i1}f_1+\cdots+a_{ir}f_r=\kappa(b_{i1}g_1+\cdots+b_{is}g_s), \qquad i=1, \cdots, r, \tag{10}$$

with a suitable factor κ and correspondingly

$$c_{j1}f_1+\cdots+c_{jr}f_r=\lambda(d_{j1}g_1+\cdots+d_{js}g_s), \qquad j=1, \cdots, s. \tag{11}$$

The functions f_i, $i=1, \cdots, r$, are linearly independent and so are the functions g_j, $j=1, \cdots, s$. (10) and (11) are not compatible unless $r=s$, $k=2r$. Introducing the notation

$$\boldsymbol{f} := \mathrm{col}\,(f_1, \cdots, f_r), \qquad \boldsymbol{g} := \mathrm{col}\,(g_1, \cdots, g_r),$$

we have

$$A\boldsymbol{f}=\kappa B\boldsymbol{g},\ C\boldsymbol{f}=\lambda D\boldsymbol{g}; \qquad \boldsymbol{f}=\kappa A^{-1}B\boldsymbol{g}=\lambda C^{-1}D\boldsymbol{g}. \tag{12}$$

Since $f_1=g_1=1$, κ and λ are constants. The fundamental system (8), i.e. the vector $\mathrm{col}\,(\boldsymbol{f}u_\alpha, \boldsymbol{g}v_\alpha)$ can be transformed into

$$\mathrm{col}\,(\boldsymbol{f}u_\alpha, \boldsymbol{f}v_\alpha) \tag{13}$$

by means of a substitution with the matrix

$$\begin{pmatrix} I & 0 \\ 0 & \kappa A^{-1}B \end{pmatrix}.$$

Next we construct the well defined linear differential equation of order r which has the functions $f_1, \cdots, f_r$ as solutions and interpret it as an equation for the solution quotients of another linear differential equation of order r (cf. Schlesinger [6] § 172). If the functions ϕ_i, $i=1, \cdots, r$, form a fundamental system for the latter equation, we have $f_i=\phi_i/\phi_1$. (9) and (10) yield

$$f_i^*(a_{11}f_1+\cdots+a_{1r}f_r)=a_{i1}f_1+\cdots+a_{ir}f_r,$$

whence

$$\phi_i^* = a_{i1}\phi_1 + \cdots + a_{ir}\phi_r, \qquad i = 1, \cdots, r.$$

We further have

$$u_\alpha^* = u_\alpha \sum_{i=1}^{r} a_{1i} f_i + \kappa v_\alpha \sum_{i=1}^{r} a_{1i} f_i$$

$$= u_\alpha \sum_{i=1}^{r} a_{1i}\phi_i/\phi_1 + \kappa v_\alpha \sum_{i=1}^{r} a_{1i}\phi_i/\phi_1,$$

$$u_\alpha^* = \frac{u_\alpha}{\phi_1}\phi_1^* + \kappa\frac{v_\alpha}{\phi_1}\phi_1^*$$

and similarly

$$v_\alpha^* = \frac{u_\alpha}{\phi_1}\phi_1^* + \lambda\frac{v_\alpha}{\phi_1}\phi_1^*$$

and we conclude that the functions ϕ_i are transformed by the matrix A and the functions

(14) $$\psi_{\alpha 1} := u_\alpha/\phi_1, \qquad \psi_{\alpha 2} := v_\alpha/\phi_1$$

by the matrix

(15) $$\begin{pmatrix} 1 & \kappa \\ 1 & \lambda \end{pmatrix}.$$

The functions (14) are solutions of an equation of order two. If the equations both for the ϕ and the ψ are known, the equation of order $k=2r$ for the components of (13) can be constructed as follows. One puts $y=\phi\psi$, writes down the derivatives y, $y', y'', \cdots, y^{(2r)}$, replaces $\phi^{(r)}, \cdots, \phi^{(2r)}$ and $\psi'', \cdots, \psi^{(2r)}$ using the differential equations and eliminates the products $\phi^{(j)}\psi$, $\phi^{(j)}\psi'$, $j=1, 2, \cdots, r-1$, from the remaining system. The originating differential equation is satisfied by all products $\phi_i\psi_{\alpha j}$, $i=1, \cdots, r$; $j=1, 2$. It has rational coefficients if this holds for the starting equations.

4. It remains to find out conditions which ensure that our differential equation of order $2r$ has polynomial solutions. Let us consider solutions of type

(16) $$y_\alpha = \psi_{\alpha 1}\phi_2 - \psi_{\alpha 2}\phi_1,$$

where $\psi_{\alpha 1}, \psi_{\alpha 2}$ are linearly independent solutions of the ψ-equation and ϕ_1, ϕ_2 suitably chosen solutions of the ϕ-equation. If x moves round a singularity, a polynomial solution of form (16) is unaltered whereas the right hand terms are transformed. After the transformation, the terms involving functions ϕ_i with $i>2$ must vanish identically. Since the ϕ_i are linearly independent, such terms must not occur at all. Therefore the substitution of the ϕ_i transforms the subspace spanned by ϕ_1 and ϕ_2 into itself just as the subspace spanned by the remaining ϕ_i. Because this is valid for all possible substitutions, one deduces that the differential equation for the ϕ_i can be decomposed and that ϕ_1 and ϕ_2 satisfy a differential equation of order two. The right hand side of (16) is transformed into

$$(a_{11}b_{21}-a_{21}b_{11})\psi_{\alpha 1}\phi_1+(a_{12}b_{22}-a_{22}b_{12})\psi_{\alpha 2}\phi_2+(a_{11}b_{22}-a_{21}b_{12})\psi_{\alpha 1}\phi_2 \\ -(a_{22}b_{11}-a_{12}b_{21})\psi_{\alpha 2}\phi_1.$$

The a_{ik} and the b_{ik} are the coefficients of the substitution which acts upon the ψ, and the ϕ respectively. The first two terms must vanish, and the cofactors of the last two terms must be equal. A short calculation yields

$$a_{ik}=b_{ik}, \qquad a_{11}a_{22}-a_{21}a_{12}=1.$$

The two fundamental systems are subjected to the same transformation. It is reasonable to assume that the ϕ_i are a special case of the $\psi_{\alpha i}$, i.e. that for fixed β

$$\phi_i(x)=\psi_{\beta i}(x), \qquad i=1, 2.$$

We define for $n=0, 1, 2, \cdots$

$$q_n(x):=\psi_{\beta+n,1}\psi_{\beta-1,2}-\psi_{\beta+n,2}\psi_{\beta-1,1}. \tag{17}$$

Regarding (3) we have

$$q_n(x)=(x-a_{\beta+n})q_{n-1}(x)-b_{\beta+n}q_{n-2}(x), \qquad n=2, 3, \cdots$$

and a comparision with (3) shows that $q_n(x)$ can be identified with $y_{\beta+n}$.

The function $q_n(x)$ satisfies a differential equation of order four by construction. It has polynomial coefficients in case this is valid for the differential equations fulfilled by the ψ_α. The expression (17) is a polynomial of degree n if and only if $q_0(x)$ is of degree zero, i.e. constant.

In order to discuss the possibility of solutions of type (7), we construct that linear differential equation of order k_1 which is satisfied by all functions $f_i u_\alpha$, $g_j v_\alpha$. If there exist solutions of type (7), we have $k_1>k$. The foregoing arguments apply to this equation, and if there exists a polynomial solution, we have $k_1=4$ and $k=3$. In this case the fundamental system is

$$u_\alpha, v_\alpha, fu_\alpha+gv_\alpha=w_\alpha.$$

Regarding (12) we find $g=(1/\kappa)f$, $w_\alpha=f(u_\alpha+(1/\kappa)v_\alpha)$, and the fundamental system can be replaced by

$$u_\alpha,\ u_\alpha+\frac{1}{\kappa}v_\alpha,\ f\left(u_\alpha+\frac{1}{\kappa}v_\alpha\right)$$

which is of type (8). Thus we are justified to exclude the possibility (7).

5. We have to investigate whether the order four is minimal. To do so we change the notation and rewrite (16) in the form

$$w_1=u_1v_2-v_1u_2. \tag{18}$$

The pairs u_1, u_2 and v_1, v_2 satisfy differential equations of order two with common singularities. The four products u_iv_j represent a fundamental system for an equation of order four. If (18) satisfies an equation of order two, the same is true for another linear combination of the products which can be written as

$$w_2=u_1v_1+cu_2v_2, \qquad c=0 \text{ or } 1,$$

by a suitable notation. Now let x move round a singularity. Due to the transformation of the pairs (u_1, u_2) and (v_1, v_2) we get

$$w_1^*=(a_{11}a_{22}-a_{21}a_{12})(u_1v_2-v_1u_2),$$
$$w_2^*=(a_{11}^2+ca_{21}^2)u_1v_1+(a_{12}^2+ca_{22}^2)u_2v_2+(a_{11}a_{12}+ca_{21}a_{22})(u_1v_2+u_2v_1).$$

Because w_2 must be a linear combination of w_1 and w_2, we obtain

$$a_{11}a_{12}+ca_{21}a_{22}=0; \qquad a_{12}^2+ca_{22}^2=c(a_{11}^2+ca_{21}^2)$$

and because of det $A=1$ we have $w_1^*=w_1$, $w_2^*=w_2$. The substitution is the identity and that for all singularities. The differential equation is degenerated, which is not possible but for the case that the equations for u_i and v_i are reducible. This case is excluded.

Now all has been proved. We summarize: In order to construct orthogonal polynomials satisfying a differential equation of order four, one has to take an irreducible differential equation of order two whose solutions depend upon a parameter α and satisfy a difference equation of type (3). One defines the function (17) and obtains polynomials if $q_0(x)$ is independent of x. Examples show that those conditions can be fulfilled (Hahn [1], Varma [8]), e.g. by the polynomial

$$q_n(x,\beta):=\frac{\Gamma(1-\beta)}{\sqrt{2\pi}}(D_{\beta+n}(x)D_{\beta-1}(-x)+(-1)^nD_{\beta+n}(-x)D_{\beta-1}(x)),$$

($D_\beta(x)$ is the parabolic cylinder function). The differential equation and the recursion formula are

$$y_n^{(4)}-(x^2-4\beta-2n)y_n''-3xy_n'+n(n+2)y_n=0,$$
$$y_n=xy_{n-1}-(n+\beta-1)y_{n-2}.$$

For $\beta=0$ we obtain the Hermite polynomials, and the differential equation is reducible in this case.

6. Let us consider the case $k=2$ somewhat in detail. For simplicity, we do not write the argument x and we denote the parameter which equals the degree of the polynomial by n. The differential equation is

$$rr_n y_n'' + s_n y_n' + t_n y_n = 0 \tag{20}$$

and in vector form with $\boldsymbol{y} := \operatorname{col}(y, y')$

$$\boldsymbol{y}_n' = A_n \boldsymbol{y}_n; \qquad A_n := \frac{1}{rr_n}\begin{pmatrix} 0 & rr_n \\ -t_n & -s_n \end{pmatrix}. \tag{21}$$

The zeros of r are the singularities of the equation. The zeros of r_n which depend upon n are nebenpoints. Equation (1) is equivalent to

$$\boldsymbol{y}_n = ((x-a_n)I + M)\boldsymbol{y}_{n-1} - b_n \boldsymbol{y}_{n-2}, \qquad M := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{22}$$

Since the differential equations for y_n and y_{n-1} have the same genuine singularities and the same substitutions, there exists a relation (cf. Schlesinger [1] § 163)

$$\boldsymbol{y}_n = P_n \boldsymbol{y}_{n-1}, \tag{23}$$

where the matrix P_n is nonsingular and has rational elements. Using this relation we can eliminate $\boldsymbol{y}_n$ and $\boldsymbol{y}_{n-2}$ from (22). The originating equation which involves only $\boldsymbol{y}_{n-1}$ holds for two linearly independent vectors and leads to the matrix equation

$$P_n = (x-a_n)I + M - b_n P_{n-1}^{-1}. \tag{24}$$

We further have from (21)

$$A_n P_n = P_n' + P_n A_{n-1}. \tag{25}$$

We put $d_n := \det P_n$ and denote the Wronskian of the distinguished fundamental system of (20) by w_n. It is divisible by r_n as well known. We learn from (23) that $w_n = d_n w_{n-1}$, and we conclude that the numerator of d_n is divisible by r_n and the denominator by r_{n-1}. (24) shows that the denominator of the elements of P_n is exactly r_{n-1} and that d_n equals r_n / r_{n-1} apart from a constant factor. The second solution of the fundamental system is defined except for a constant which we choose such that

$$d_n = \frac{b_{n+1} r_n}{r_{n-1}}. \tag{26}$$

We write

$$P_n = \frac{1}{r_{n-1}} \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}.$$

The elements of the matrix are polynomials. The relation (24) furnishes us with the four equations

$$(27) \qquad \alpha_n = (x - a_n) r_{n-1} - \delta_{n-1}; \qquad \beta_n = \beta_{n-1},$$

$$(28) \qquad \gamma_n = r_{n-1} + \gamma_{n-1}; \qquad \delta_n = (x - a_n) r_{n-1} - \alpha_{n-1}.$$

Obviously

$$(29) \qquad \beta_n =: \beta \quad \text{and} \quad \alpha_n - \delta_n =: q$$

are independent of n. The equations for the elements p_{12} and p_{21} in (25) are

$$r r_{n-1} \delta_n = -r r'_{n-1} \beta + r r_{n-1} \beta' + r r_{n-1} \alpha_n - \beta s_{n-1},$$
$$\beta t_n = r(r_{n-1} \alpha'_n - r'_{n-1} \alpha_n - r_{n-1} \gamma_n).$$

We see that β is divisible by r, $\beta =: rp$, and that

$$(30) \qquad s_{n-1} = -r r'_{n-1} + \frac{q + (rp)'}{p} r_{n-1}.$$

Combining (27) and (28) we obtain

$$(31) \qquad \delta_n + \delta_{n-1} = (x - a_n) r_{n-1} - q.$$

Calculating d_n immediately and regarding (26) we find

$$\alpha_n \delta_n = b_{n+1} r_n r_{n-1} + r p \gamma_n.$$

We now calculate $\alpha_n \delta_n - \alpha_{n-1} \delta_{n-1}$, replace $\gamma_n - \gamma_{n-1}$ by r_{n-1} according to (28) and eliminate α_n and α_{n-1} by means of (27) and (28). We obtain

$$(32) \qquad (\delta_n - \delta_{n-1})(x - a_n) = b_{n+1} r_n - b_n r_{n-2} - rp.$$

The equations (31) and (32) can be solved for δ_n and δ_{n-1} and then we can put up an equation which does no longer contain the greek letter polynomials. This equation is an identity in x and equivalent to a system of nonlinear difference equations for the quantities a_n, b_n, and the coefficients of r_n. The coefficients of p, q, and r are parameters. The difference equations are necessary conditions for the said quantities which have to be completed by additional requirements, e.g. the polynomial $q + (rp)'$, occurring in (30), must be divisible by p.

7. We finally mention a procedure which furnishes us with polynomials of the desired type. If we have a sequence $\{y_n\}$ of polynomials satisfying (1) with $\alpha = n$ and if we define a sequence $\{c_n\}$ of numbers by the difference equation

$$c_n c_{n-1} + b_n = a_n c_n,$$

then the polynomials $w_n := y_n + c_n y_{n-1}$ fulfil a recursion formula of type (1). If the y_n satisfy a differential equation (20), the same is true for the w_n (Hahn [3]).

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