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On Differentiability of the Minimal Time Function

By

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1. Introduction, notation, summary.

The aims of this paper are paradoxical. In one direction we study and apply differentiability of minimal time function (see subsequent definitions and notation): Theorems 1, 5 and 2, 3, 4, 6. The reason for this interest is in the familiar construction of feed-back controls [14, p. 146], but this will not be treated here.

In the other, we show that often the minimal time function fails to have reasonable differentiability properties: Corollaries 1 and 2 to Lemma 2, Proposition 4, Lemma 3 and its corollaries, Lemma 7. One of the points is that, in the better examples, the dimension of control constraint space is less than that of state space, so that the constraint set cannot be a neighborhood of the origin (see [13], p. 574, and Theorems 3, 4 in [9]). This seems to indicate that direct application of the Bellman equation, and of the corresponding Isaacs equation in value-centered treatment of game theory, is confined to the fixed time-interval case (or to very artifical constraint sets).

Throughout this paper we will be concerned with the control system, in Euclidean n-space R^n, symbolically represented by

$$\dot{x}=Ax-u, \qquad u(t)\in U,$$

under assumptions listed below. (The unorthodox minus sign at u is the right choice in those cases where it matters, i.e., for U not symmetric about 0.)

The problem is, implicitly, that of reaching the origin in least time. We will study the minimal time function T defined by

$$T(x)=\inf\{t: x\in R(t)\}. \tag{1}$$

Here $R(t)$ is the reachable set at time $t\geq 0$,

$$R(t)=\left\{\int_0^t e^{-As}u(s)ds: \text{measurable } u: [0, t]\to U\right\}$$

(the Aumann integral of $e^{-As}U$). We will take T finite-valued, i.e., consider only

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$T: R \to R^1$, where $R = \bigcup_{t \geq 0} R(t)$ is the reachable set (note that otherwise (1) defines $T(x) = +\infty$ for $x \notin R$).

Assumptions. The dimension n of state space is at least 1; the coefficient matrix A is real n-square; the constraint set U is compact, convex, and contains the origin; the control system is *proper* in the sense that $R(t)$ is a neighborhood of 0 for each $t > 0$.

Some remarks about these. $n = 0$ occurs in decomposition theorems; $n \geq 2$ was needed in [7]. It is a famous result that convexity of U may be omitted. The definition of proper systems is a modification of one due to LaSalle, [13]. It implies controllability (i.e., that R is a neighborhood of 0); if U is symmetric about 0, it is equivalent with controllability, but not otherwise. Even for asymmetric constraint sets there is a working characterization of controllability (Brammer [2]; also [8]); For properness there are reasonable sufficient, and also necessary, conditions; a rather brutal one is that the constraint U be a neighborhood of the origin.

It is well known that the sets $R(t)$ are convex and compact, and that, in (1), the infimum is actually a minimum. Thus $R(t) = \{x : T(x) \leq t\}$; under our assumptions that the system is proper, we even have

$$\partial R(t) = \{x : T(x) = t\} \tag{2}$$

(for controllable systems which are not proper, the t-isochrone is a proper subset of the boundary). Finally, T is continuous: see Theorem 1 in [6], with unnecessary assumptions on U; in point of fact, T is continuous (or merely at 0) if and only if the system is proper.

From Section 3 onward we will need the concept of support function δ, to a compact, convex neighborhood U of 0 (Minkowski's **distanz funktion** [15]; also see [5]). For any $x \neq 0$ define $\delta(x) = \delta$ by requiring

$$0 < \delta < +\infty, \qquad \frac{x}{\delta} \in \partial U,$$

and also set $\delta(0) = 0$. It is easily verified that δ is subadditive and homogeneous; it follows that δ is a Lipschitz function (ibid. p. 133), since

$$|\delta(x) - \delta(y)| \leq \max\,(\delta(x-y),\ \delta(y-x)) \leq |x-y| \cdot \max_{|z|=1} \delta(z),$$

with the maximum finite.

The notation $DT(x)$ will be used for the gradient (Jacobian row-vector) of T at x.

The standing assumption and results just mentioned (in particular (2)) will be used without explicit reference.

Section 2 contains the basic apparatus. This consists of the limit theorem, Proposition 1, which is an expansion of a very special case of a result due to Aumann (the fundamental theorem of the integral calculus of set-valued functions). It has been used, for other purposes, in [3] and [4]. The second is the addition formula (6) for reachable sets, re-interpreted in terms of the minimal time function, Proposition 2. The addition formula itself should probably be classified as good folk-lore (e.g., one of the inclusions is the "elementary observation" in Corollary 15 of [6]); it has been considerably exploited, in [12]. The last, Proposition 3, properly belongs to calculus of several variables; the assertion is, essentially, that the gradient is not only the only normal (differential geometry) but also the only exterior normal (convex set theory).

Some direct consequences are presented in Section 3. In particular, it is shown that T never has a differential at 0, but does have directional derivatives there if the constraint set is a neighborhood of 0. It may be noted how, in several instances, Proposition 2 is used to carry over information about local behavior at $x=0$ to other points x. This method appears to fail for Theorem 1 (also see the corollary), leading to

Conjecture 1. *If U is a neighborhood of the origin, then T has directional derivatives at each point $x \in R$.*

In Section 4, Theorem 2, we present an apparently new principle of maximality for T. Some of the consequences then are given in the last section.

It is interesting to list the maximality principles now available (assumptions are omitted; we refer to our notation and control system). From Pontrjagin's maximum principle (see Theorem 4),

$$\max_{y \in R(t)} DT(x)y = DT(x)\cdot x \qquad (t=T(x));$$

from Bellman's principle of optimality (see [13, p. 146] and our Theorem 3),

$$\max_{u \in U} DT(x)u = 1 + DT(x)\cdot Ax;$$

Theorem 2 states that

$$\max_{u \in U} DT(x)e^{-AT(x)}u = 1$$

and Theorem 6 that

$$\max_{x \in \partial R(t)} DT(x)e^{At}u = 1 \qquad (u \in \partial U).$$

Curiously enough, for Pontrjagin's principle, both the basis and the geometric interpretation are obvious; for Bellman's principle, the basis is optimality, but the geometry is elusive; and for the last two, the geometric facts are clear, Lemma 8, but the underlying principles are not.

Finally, a mention of what this paper should, but does not contain. The first of these is a proof for

Conjecture 2. *If the support function* δ *is continuously differentiable outside the origin, then* T *is* C^1 *in* $R-\{0\}$.

According to Theorem 5, one need only prove differentiability. If one seeks a condition on U independent of A, then $\delta \in C^1$ outside 0 is necessary (take $A=0$, whereupon $T=\delta$).

Conceivably one might then approximate U, e.g. from the outside, by constraint sets U_k having smooth support functions; it would be desirable to study convergence of the corresponding DT_k (and, possibly, of the feedback controls obtained thence). This could re-kindle hopes of using Hamilton-Jacobi equations of approximating systems in time-optimal problems of control and game theory.

2. Basic apparatus.

Proposition 1. (The limit theorem.) *We have*

$$\lim_{t\to 0} \frac{R(t)}{t} = U, \tag{3}$$

$$\lim_{t\to 0} \frac{\partial R(t)}{t} = \partial U = \lim_{x\to 0} \left\{\frac{x}{T(x)}\right\}. \tag{4}$$

(*All limits of sets—even the singletons in* (4)*—are taken relative to the Hausdorff metric.*)

Proof. We begin by establishing that, as $t\to 0$,

$$\int_0^t e^{-As}ds\cdot U \subset R(t) \subset \int_0^t e^{-As}ds\cdot U + \mathcal{O}(t^2). \tag{5}$$

This estimates the difference between $R(t)$ and $\int_e^t e^{-As}ds\cdot U$ (i.e., all admissible controls versus the constant controls). The first inclusion is immediate. For the second, take any $t>0$ and $x \in R(t)$, so that

$$x = \int_0^t e^{-As}u(s)ds, \qquad u\colon [0, t]\to U.$$

Set

$$v=\frac{1}{t}\int_0^t u(s)ds, \qquad y=\int_0^t e^{-As}ds\ v.$$

Here $v \in U$ since U is closed and convex; thus it will be sufficient to show that $x-y=\mathcal{O}(t^2)$, estimate independent of x. We have $e=I+(e-I)$, so that

$$x-y=\int_0^t (u(s)-v)ds+\int_0^t (e^{-As}-I)(u(s)-v)ds;$$

the first term vanishes, by choice of v; and the second has norm at most

$$\int_0^t |e^{-As}-I|\ ds\cdot \text{diam}\ U=\mathcal{O}(t^2).$$

Having (5), divide by t and take $t\to 0$ to obtain (3).

For $\lim_{t\to 0} \partial R(t)/t=\partial U$ we need to verify two facts. First, that ∂U is contained in the lim inf, or, equivalently, that

$$\text{dist}\left(x, \frac{\partial R(t)}{t}\right)\to 0 \quad \text{for each } x \in \partial U.$$

Choose $y \notin U$ close to x; by (3), $y \notin R(t)/t$ for small t, so that y has the same distance from $R(t)/t$ as from its boundary. Thus

$$\begin{aligned}\text{dist}\left(x, \frac{\partial R(t)}{t}\right)&\leq |x-y|+\text{dist}\left(y, \frac{\partial R(t)}{t}\right)\\ &=|x-y|+\text{dist}\left(y, \frac{R(t)}{t}\right)\leq 2\,|x-y|+\text{dist}\left(x, \frac{R(t)}{t}\right)\end{aligned}$$

where the last term tends to 0 with t according to (3), and the first can be made arbitrarily small (this part did not need convexity). Second, that ∂U contains the lim sup, i.e., that, if

$$\frac{\partial R(t_k)}{t_k} \ni x_k \to x, \qquad t_k\to 0,$$

then $x \in \partial U$. For this consider unit exterior normals c_k to $R(t_k)/t_k$ at x_k; it is easily verified that any $c=\lim c_{k_m}$ is an exterior normal to U at x, using (3).

To complete (4), observe that

$$\frac{\partial R(t)}{t}=\left\{\frac{T(x)}{x}: T(x)=t\right\}$$

and that $x\to 0$ is equivalent to $T(x)\to 0$.

Corollary. dist $(R(t), tU) = \mathcal{O}(t^2)$ *as* $t \to 0$.

Remarks. Both (3) and the first formula in (4) hold even without assuming that the system is proper; furthermore, the boundaries can even be taken within the controllability subspace.

An alternate proof of (3) can be obtained from a far more general theorem due to Aumann [1]; in our autonomous case it specializes to

$$\lim_{t\to\theta} \frac{1}{t-\theta}\left\{\int_\theta^t e^{-As}u(s)ds : \text{all } u(s) \in U\right\} = e^{-A\theta}U$$

for almost all θ; using continuity of $e^{-A\theta}$, this extends to *all* θ, so that we may take $\theta = 0$. Apparently there is a continuous version of Aumann's theorem.

In the context of the corollary, one-sided second order estimates are available:

$$R(t) \subset tU - \frac{t^2}{2}A\cdot U + \mathcal{O}(t^3),$$

$$\left(t\cdot I - \frac{t^2}{2}A\right)U \subset R(t) + \mathcal{O}(t^3).$$

Lemma 1. *For any nonnegative* t, s *we have the addition formula*

$$R(t+s) = R(t) + e^{-At}R(s). \tag{6}$$

Both inclusions here are immediate, using

$$\int_t^{t+s} e^{-A\sigma}u(\sigma)d\sigma = e^{-At}\int_0^s e^{-A\sigma}u(t+\sigma)d\sigma;$$

the essential point is that the constraint set U is independent of time; actually, no other assumptions on U are needed. (In point of fact, there is an almost immediate version of the addition formula applying to non-autonomous linear systems.)

Proposition 2. *Let* $x, y \in R$, $t = T(x)$; *then*

$$T(x + e^{-At}y) \leq T(x) + T(y).$$

Furthermore, equality holds for any $x \in R$, $s \geq 0$ *and some* y *with* $T(y) = s$; *and also for any* $y \in R$, $t \geq 0$ *and some* x *with* $T(x) = t$.

Proof. The inequality is a rephrasing of one inclusion in the addition formula. The remaining assertions are also obtained thence, by convex trickery, and we will only prove the first.

Since $t = T(x)$, we have $x \in \partial R(t)$, and therefore can find a nonzero exterior normal c to $R(t)$ at x.

Next choose a point $y \in R(s)$ such that c is also an exterior normal to $e^{-At}R(s)$ at $e^{-At}y$; necessarily $T(y)=s$. But then c is an exterior normal, at $z=x+e^{-At}y$, to (6); hence $T(z)=t+s=T(x)+T(y)$.

Proposition 3. *Assume* $f: R^n \to R^1$ *has* $M=\{x: f(x)\leq\alpha\}$ *convex for some* α. *If* f *has a differential at a boundary point* x *of* M *and* $Df(x)\neq 0$, *then* $Df(x)/|Df(x)|$ *is the only unit exterior normal to* M *at* x.

Proof. For convenience we assume $x=0$, $\alpha=0=f(x)$, and write $d'=Df(0)$. Consider any exterior normal c to M at 0. It will be sufficient to show that

$$(7) \qquad d'x<0 \quad \text{implies} \quad c'x\leq 0.$$

Indeed, a limit argument, using $d\neq 0$, will then yield that even $d'y\leq 0$ implies $c'y \leq 0$, so that $c=\lambda d$ for some $\lambda\geq 0$ (a rudimentary version of Farkas' lemma).

First let ρ be the remainder term in

$$f(x)=f(x)-f(0)=d'x+\rho(x), \quad \rho(x)=o(|x|) \qquad \text{as } x\to 0.$$

To prove (7), assume $d'x<0$, so that also $d'y<0$ for $y=x/|x|$. For small $\varepsilon>0$ we then have

$$0>d'y+\frac{\rho(\varepsilon y)}{\varepsilon}=\frac{1}{\varepsilon}(d'\varepsilon y+\rho(\varepsilon y))=\frac{f(\varepsilon y)}{\varepsilon},$$

so that $f(\varepsilon y)<0$ and $\varepsilon y \in M$ for small $\varepsilon>0$. By assumption on c, $c'\varepsilon y\leq 0$; but this then holds for all $\varepsilon>0$, in particular, for $\varepsilon=|x|$; thus indeed $c'x\leq 0$ as it was required to prove.

Remark. The convexity assumption may be removed if exterior normals are replaced by the following concept: vectors c such that

$$\limsup_{M\ni y\to x} c'\frac{y-x}{|y-x|}\leq 0.$$

Differentiability may be weakened slightly to: f has a directional derivative which is linear. However, $Df(x)\neq 0$ cannot be omitted: in R^2, if $f(x,y)=y^2$, then $\{f\leq 0\}$ is the x-axis, with two unit exterior normals at the origin.

3. Local behavior.

The following result will be improved upon in Lemma 4.

Lemma 2. *T has no stationary points* (*i.e.*, *points where its differential vanishes*).

Proof. Note first that $T(y)/|y|$ is bounded away from 0 as $y \to 0$, since its reciprocal has

$$\limsup_{y\to 0} \frac{|y|}{T(y)} = \max_{u\in\partial U} |u| < +\infty$$

by (4) in Proposition 1.

Now assume T is differentiable at x and $DT(x)=0$; equivalently, $T(x+z)-T(x)=o(|z|)$ as $z\to 0$. Set $t=T(x)$; for each $s>0$ choose y so that equality holds in Proposition 2; then $y\to 0$ with s, and

$$0 \leftarrow \frac{T(x+e^{-At}y)-T(x)}{|e^{-At}y|} = \frac{T(y)}{|y|}\cdot\frac{|y|}{|e^{-At}y|} .$$

But the first factor is bounded away from 0, and, obviously, the second also: contradiction.

Corollary 1. *T does not have a differential at* 0.

Proof. T has a minimum there, so its differential would have to vanish.

Corollary 2. *If U has an n-dimensional corner, at a point $u \in U$, then T does not have a differential at any point of the analytic curve $\left\{\int_0^t e^{-As}ds\, u : 0\le t\le \varepsilon\right\}$ for small enough $\varepsilon>0$.*

Proof. The set of exterior normals to U at u has nonempty interior: some normals remain such even if changed slightly. Thus there exists $\varepsilon>0$ and distinct unit vectors c_1, c_2 such that $e^{-A's}c_k$ is an exterior normal to U at u for $k=1, 2$ and $s\in[0,\varepsilon]$. For any $t\in[0,\varepsilon]$ consider the point $x=\int_0^t e^{-As}ds\, u$ on the described curve (obviously $x\in R(t)$). For any

$$y\in R(t),\quad y=\int_0^t e^{-As}v(s)ds, \qquad v\colon [0,t]\to U,$$

we have

$$c_k' e^{-As}v(s) \le c_k' e^{-As}u, \qquad c_k' y \le c_k' x;$$

thus both c_k are exterior normals to $R(t)$ at x. Thus there are distinct unit exterior normals at x.

Now, if T had a differential at x, necessarily $DT(x)\ne 0$ by the preceding lemma; and we obtain a contradiction with Proposition 3.

Corollary 3. *A necessary condition that T have a differential at all points of $\partial R(t)$ for given $t>0$ is that $R(t)$ have no corners.*

(*Proof*: Lemma 2 and Proposition 3.)

Proposition 4. *If, and only if, U is a neighborhood of 0, then $T(x)=\mathcal{O}(|x|)$ as $x\to 0$. In the positive case T is locally a Lipschitz function in R; in particular, it has a differential almost everywhere in R.*

Proof. Again from (4), $\liminf |x|/T(x)=\min\{|u|: u\in\partial U\}$, and the latter is strictly positive precisely when $0\notin\partial U$, i.e., U is a neighborhood of 0.

Assuming this, we have $T(x)\leq\alpha|x|$ for $|x|\leq\beta$ (and some α,β); from Proposition 2, with notation changed,

$$T(y)-T(x)\leq T(e^{At}(y-x)) \qquad (t=T(x)).$$

Thus, if x, y are in $R(\theta)$ and $e^{At}(y-x)$ has length $\leq\beta$, then

$$T(y)-T(x)\leq\alpha|e^{At}(y-x)|\leq\alpha e^{|A|\theta}|y-x|.$$

Since the latter estimate is symmetric between x and y, we have

$$\frac{|T(y)-T(x)|}{|y-x|}\leq\alpha e^{|A|\theta}$$

for x, y in $R(\theta)$ if $|e^{At}(y-x)|\leq\beta$.

The last assertion is a classical result [16, p. 311].

This extends Proposition 14 and Theorem 16 of [6] to more general constraint sets. For the case that the constraint set is not a neighborhood of 0, we can provide more information:

Lemma 3. *If, for an $m\geq 0$, the set*

$$U-AU+A^2U-\cdots+(-1)^mA^mU$$

is not a neighborhood of 0, then there exists a unit vector y such that

$$\limsup_{\alpha\to 0}\frac{T(\alpha y)}{\alpha^\mu}>0 \qquad \textit{for } \mu=\frac{1}{m+2}.$$

Proof. Choose y as an exterior normal at 0 to the indicated convex set. Then also $y'(-A)^ku\leq 0$ for $k=0,\cdots,m$ and all $u\in U$. For $\alpha>0$ set $x=\alpha y$. Now either $T(x)=+\infty$, and the subsequent argument becomes trivial, or $t=T(x)<+\infty$. In this case

$$x=\int_0^t e^{-As}u(s)ds, \qquad u\colon [0,t]\to U,$$

$$|x|=\alpha=y'x=\sum_{k=0}^{m}\int_0^t y'(-A)^k u(s)\frac{s^k}{k!}ds+\sum_{k>m}\int_0^t \leq 0+\mathcal{O}(t^{m+2}),$$

so that, with $\mu=1/(m+2)$,

$$\frac{T(\alpha y)}{\alpha^\mu}=\frac{t}{|x|^\mu}>\varepsilon>0$$

for an ε independent of α.

Corollary 1. *If U is a one-dimensional segment, then*

$$\limsup_{x\to 0} T(x)/|x|^\nu=+\infty \qquad \text{for all } \nu>1/n.$$

Corollary 2. *Under the assumptions of the lemma, for every $t\geq 0$ there exists a unit vector y such that, for $\mu=1/(m+2)$,*

$$\limsup_{\alpha\to 0}\sup_{x\in\partial R(t)}\frac{T(x+\alpha y)-T(x)}{\alpha^\mu}>0.$$

Proof. Let y be the unit vector in the direction $e^{-At}z$, where z is an exterior normal at 0 to the indicated convex set. For every α we then choose $x\in\partial R(t)$ so that

$$T(x+\alpha y)-T(x)=T(\alpha e^{At}z)$$

(see Proposition 2); finally we apply Lemma 3.

Next we show, using the support function δ of U, that T has directional derivatives at 0.

Theorem 1. *If U is a neighborhood of 0, then, for every x,*

$$\lim_{\alpha\to 0+}\frac{T(\alpha x)}{\alpha}=\delta(x).$$

Proof. We need only treat $x\neq 0$. For small $|x|$ we have $x\in\partial R(T(x))$. Thus, for any $x\neq 0$ and small enough $\alpha>0$,

$$\frac{\alpha}{T(\alpha x)}\cdot x\in\frac{\partial R(T(\alpha x))}{T(\alpha x)}.$$

Take any sequence $\alpha_k\to 0+$; some subsequence $\alpha_{k_j}=\alpha$ will have $\frac{\alpha}{T(\alpha)}\cdot x\to u\in\partial U$

by Proposition 1. Since $u \neq 0$, the positive factor $\alpha / T(\alpha x)$ has a finite limit β, and $\beta x \in \partial U$. By definition, $\beta = 1/\delta(x)$, and this is independent of the chosen sequence α_k. Therefore

$$\lim_{\alpha \to 0+} \frac{\alpha}{T(\alpha x)} = \frac{1}{\delta(x)}$$

as asserted.

Remarks. Somewhat more can be proved by a like reasoning: that $\delta(x)/T(x) = 1 + \mathcal{O}(|x|)$; in particular, in Theorem 1, convergence is uniform on every ball (this can also be verified directly). Hence, if T_A corresponds to coefficient matrix A, and T_B to B, with U preserved, then $T_A / T_B \to 1$ as $x \to 0$. In the same situation we conclude from Lemma 6 (to follow) that $\frac{DT_A(x) \cdot x}{DT_B(x)x} \to 1$ as $x \to 0$,

Corollary. *For any $x \in R$ and y,*

$$\limsup_{\alpha \to 0+} \frac{T(x+\alpha y) - T(x)}{\alpha} \leq \delta(e^{AT(x)} y) \leq |e^{AT(x)}|\,|y| \max_{|z|=1} \delta(z)$$

$$\liminf_{\alpha \to 0+} \frac{T(x+\alpha y) - T(x)}{\alpha} \geq -\delta(-e^{AT(x)} y).$$

Proof. Apply the theorem after first estimating via Proposition 2; for the lower estimate write

$$T(x+\alpha y) - T(x) = -(T((x+\alpha y) - \alpha y) - T(x+\alpha y))$$

and use $\exp AT(x+\alpha y) \to \exp AT(x)$.

4. Maximality principles.

Theorem 2. *If T has a differential at x, then*

$$\max_{u \in U} DT(x) e^{-AT(x)} u = 1.$$

Proof. Write $t = T(x)$. For any y in R we have

$$T(x + e^{-At} y) - T(x) = DT(x) e^{-At} y + o(|y|);$$

apply the estimate of Proposition 2, obtaining, for $y \neq 0$,

$$DT(x) e^{-At} \frac{y}{T(y)} \leq 1 + o(1) \cdot \frac{|y|}{T(y)}. \tag{8}$$

If we take $y \to 0$, then the last term tends to 0 $\left(\text{since then } T(y)=s\to 0, \text{ and } \frac{y}{s} \in \frac{\partial R(s)}{s} \to \partial U \text{ by Proposition 1}\right)$.

According to Proposition 1, every $u \in \partial U$ is a limit of suitable $y/T(y)$ with $y \to 0$. Hence

$$DT(x)e^{-At}u \leq 1 \tag{9}$$

for each $u \in \partial U$, and, therefore, for $u \in U$ also. Second, for each $s>0$ we can find $y \in \partial R(s)$ (thus, $s=T(y)$) so that equality holds in the estimate; then equality will hold in (8), and a suitable subsequence of y/s will converge to a point $u \in \partial U$ (Proposition 1); therefore equality holds in (9) for this particular u.

There is a companion formula in Theorem 5; the geometric interpretations of these appears in Lemma 8.

The following formula, with notational changes, is well known [14, p. 146] under the assumption that T is continuously differentiable in a neighborhood of x. The strength of Proposition 1 is illustrated by weakening the hypothesis. The author has contributed to (bad) folk-lore by inadvertently asserting the stronger version without proof in [6, p. 340].

Theorem 3. *If T has a differential at x, then*

$$\max_{u \in U} DT(x)u = 1 + DT(x)\cdot Ax.$$

Proof. Consider any admissible control $u: R^+ \to U$; then, by the principle of optimality, the point $x(t)=e^{At}\left(x-\int_0^t e^{-As}u(s)ds\right)$ has

$$T(x(t)) \geq T(x)-t \qquad (0<t\leq T(x))$$

with equality if u is taken optimally. Therefore

$$-1 \leq \frac{T(x(t))-T(x)}{t} = DT(x)\cdot\frac{x(t)-x}{t} + \frac{o(|x(t)-x|)}{t}. \tag{10}$$

Here the term

$$\begin{aligned} \frac{x(t)-x}{t} &= \frac{e^{At}-I}{t}x - e^{At}\frac{1}{t}\int_0^t e^{-As}u(s)ds \\ &\in \frac{e^{At}-I}{t}x - e^{At}\frac{R(t)}{t}; \end{aligned} \tag{11}$$

according to Proposition 1, then, it remains bounded as $t \to 0$. Hence the remainder term in (10) is

$$o(1) \cdot \frac{|x(t)-x|}{t} \to 0 \qquad \text{as } t \to 0.$$

Returning to (11), there is a sequence $t = t_k \to 0$ such that

$$\frac{x(t)-x}{t} \to Ax - u_0, \qquad u_0 \in U$$

(see Proposition 1), and any $u_0 \in U$ can be so obtained by appropriate choice of u (loc. cit.). Thus, from (10),

$$-1 \leq DT(x)(Ax-u_0) \qquad (\text{all } u_0 \in U),$$

with equality attained by optimal u on taking appropriate $t_k \to 0$.

Remark. The author has the impression that the maximum principles described in the two preceding theorems are, in some sense, independent, and even complementary.

Assume that U has only finitely many extreme points $u_1, \cdots, u_r$; the maximality relations of Theorems 2 and 3 will probably reduce, locally, to simultaneous partial differential equations for T,

$$DT(x)e^{-AT(x)}u_k = 1, \qquad DT(x)(u_l - Ax) = 1$$

(quasi-linear and linear, respectively); and Theorem 1 will act as a boundary condition. E.g. for $n=2$ this might determine T completely, see [*].

Theorem 4. *If T has a differential at x, and we set $t = T(x)$, then $DT(x)$ is an exterior normal to $R(t)$ at x, i.e.,*

$$\max_{y \in R(t)} DT(x)y = DT(x)x.$$

Proof. We have $x \in \partial R(t)$. There exists a unit exterior normal c to $R(t)$ at x (this is what the Pontrjagin maximum principle reduces to in the linear time-optimal problem). The assertion follows on applying Lemma 2 and Proposition 3.

5. Some consequences.

Lemma 4. $DT \cdot e^{-AT}$ *is bounded away from* 0 (*for points at which T has a differential*).

Proof. Consider a ball, centered at 0, and with radius $\rho > 0$ so large that it contains U. Then

$$\rho\,|DTe^{-AT}| = \max_{|v|\le\rho} DTe^{-AT}v \ge \max_{u\in U} DTe^{-AT}u = 1$$

by Theorem 2; thus $|DTe^{-AT}| \ge \rho^{-1}$.

Lemma 5. *If U is a neighborhood of* 0, *then $DT\cdot e^{-AT}$ is bounded (for points at which T has a differential).*

This is established similarly, estimating U from below.

Remark. We have used Theorem 2; Theorem 3 yields other estimates: if U contains a ball of radius σ, and is in one of radius ρ (both centered at the origin), then

$$\frac{1}{\rho+|Ax|} \le |DT(x)| \le \frac{1}{\sigma-|Ax|}$$

(the latter for $|Ax|<\sigma$ only).

Lemma 6. *If U is a neighborhood of* 0, *then*

$$DT(x)\cdot\frac{x}{T(x)} = 1 + \mathcal{O}(T(x)) \qquad \text{as } x\to 0.$$

Proof. Set $t=T(x)$. Since $d'=DT(x)$ is an exterior normal to $R(t)$ at x by Theorem 4,

$$d'\cdot\frac{x}{t} = \max_{y\in R(t)} d'\cdot\frac{y}{t} = \max_{u:R^1\to U} d'\cdot e^{-At}\cdot\frac{1}{t}\int_0^t e^{As}u(s)ds, \tag{12}$$

where we have written

$$y = \int_0^t e^{-As}v(s)ds = e^{-At}\int_0^t e^{As}v(t-s)ds.$$

According to the corollary to Proposition 1 (with $-A$ replacing A), the integral term in (12) is

$$\int_0^t e^{As}u(s)ds \in t\cdot U + \mathcal{O}(t^2).$$

Since DT is bounded near 0 (Lemma 5), we have in (12)

$$\frac{d'x}{t} = \max_{u\in U} d'e^{-At}u + \mathcal{O}(t) = 1 + \mathcal{O}(t).$$

Remark. Neither of the preceding two results holds for $\ddot{x}=u$, $|u(t)|\leq 1$: the assumption on U cannot be omitted.

Theorem 5. *DT is continuous on the set of points at which T has a differential.*

Proof. First note that $DT(x)\neq 0$ at such points, by Lemma 2. Now assume T has a differential at x_k and x, and $x_k\to x$; set $t_k=T(x_k)$, $t=T(x)$, so that $t_k\to t$. Then $DT(x_k)/|DT(x_k)|$ is a unit exterior normal to $R(t_k)$ (Theorem 4) at x_k, so that each limit of a subsequence is a unit exterior normal to $R(t)$ at x. By Proposition 3, this must coincide with $DT(x)/|DT(x)|$. Thus

$$DT(x_k)\to\lambda DT(x)$$

for some $\lambda\geq 0$ and subsequence, and it only remains to show that necessarily $\lambda=1$. Apply Theorem 2: choose $u_k\in U$ with $DT(x_k)e^{-At_k}u_k=1$, and a convergent subsequence $u_k\to u\in U$. Taking limits, we find

$$1=\lambda DT(x)e^{-At}u\leq\lambda\cdot 1.$$

For the opposite inequality take $u\in U$ with $DT(x)e^{-At}u=1$; then, for small $\varepsilon>0$ and large k,

$$1-\varepsilon<\frac{1}{\lambda}DT(x_k)e^{-At_k}u\leq\frac{1}{\lambda}\cdot 1,$$

so that $\lambda<1/1-\varepsilon$; finally, let $\varepsilon\to 0$.

Lemma 7. *Assume U is not a neighborhood of* 0. *If* $c\neq 0$ *is an exterior normal to U at* 0, *let* $x=x_t$ *be a point of* $R(t)$ *at which* $e^{A't}c$ *is an exterior normal. Conclusion: T does not have a differential at x. Furthermore, either T does not have a differential in a neighborhood of x, or* $DT(y)\to\infty$ *as* $y\to x$, *or* $R(t)$ *has a corner at x.*

Proof. If T had a differential at x, then $d'=DT(x)$ would have the same direction as $(e^{A't}c)'$ (Proposition 3, Lemma 2, Theorem 4); thus $d'=\alpha c'e^{At}$, $\alpha\geq 0$. But then

$$d'e^{-At}u=\alpha c'u\leq 0 \qquad (u\in U)$$

by assumption on c, contradicting Theorem 2.

In the second assertion, assume the first two alternatives do not obtain. Thus there exist $y=y_k\to x$ such that $DT(y)$ exist and converge to some finite point d. For each k find $u=u_k\in U$ so that $DT(y)e^{-AT(y)}u=1$ (Theorem 2); we may assume

$u_k \to u_0 \in U$. Then $d'e^{-At}u_0 = 1$, d is an exterior normal to $R(t)$ at x (as a limit of exterior normals), and d is not parallel to $e^{A't}c$ since

$$0 \geq c'u = (e^{A't}c)'e^{-At}u \qquad \text{for } u \in U.$$

Thus $R(t)$ has distinct exterior normal directions at x.

As an application, consider the control system in R^2 associated with the harmonic oscillator, $\ddot{x} + x = u$, $|u(t)| \leq 1$. It is known that T is continuously differentiable in an open, dense subset of $R(\pi)$ [7, Theorem 6], and that $R(\pi)$ has no corners [11, Exercise 15.2]. Hence, at points with coordinates $(\pm 2, 0)$ it is the second alternative that obtains.

If T were differentiable at 0, then a simple argument using Proposition 2 would show that $DT(x)e^{-AT(x)} = DT(0)$; however, Corollary 1 to Lemma 2 disposes of this. One might conjecture that, at least, $DT(x)e^{-AT(x)}$ is independent of x; this is readily disproved by elementary examples. However, we will show that the set

$$\{DT(x) : x \in \partial R(t)\} \cdot e^{-At} \tag{13}$$

is independent of t. Recall Minkowski's concept of the polar E^* to a subset $E \subset R^n$ [15], [5]:

$$E^* = \{y : y'x \leq 1 \text{ for all } x \in E\}.$$

Obviously E^* is convex and closed; it is readily proved that $E^{**} = E$ if E is convex and closed.

Lemma 8. *If T has a differential at all points of $\partial R(t)$, then*

$$U^* = \{v : v' = \alpha DT(x)e^{-At},\ 0 \leq \alpha \leq 1,\ x \in \partial R(t)\}. \tag{14}$$

Proof. Denote the set on the right by V. Then $V \subset U^*$ follows from Theorem 2 (and $0 \leq \alpha \leq 1$). Conversely, assume $v \in U^*$; in proving $v \in V$ we need only treat $v \neq 0$. Find a point x at which $e^{A't}v$ is an exterior normal to $R(t)$; by Proposition 3 and Lemma 2,

$$(e^{A't}v)' = v'e^{At} = \alpha DT(x),$$

for some $\alpha > 0$. Hence

$$1 \geq \max_{u \in V} v'u = \alpha \max_{u} DT(x)e^{-At}u = \alpha,$$

by Theorem 2. Thus $v' = \alpha DT(x)e^{-At}$ and $0 \leq \alpha \leq 1$, i.e., $v \in V$.

Corollary. *If T has a differential at all points of $\partial R(t)$, then the set* (13) *is the boundary of a compact and convex neighborhood of* 0, *independent of t.*

Proof. From the assumptions and Lemma 7, U is a neighborhood of 0. Hence U^* is compact, convex and a neighborhood of 0, so that $\alpha=1$ in (14) defines its boundary.

Theorem 5. *If T has a differential at all points of $\partial R(t)$, then, for every $u \in \partial U$,*

$$\max_{x \in \partial R(t)} DT(x)e^{-At}u=1.$$

Proof. Using the notation of the proof of Lemma 8, $U^*=V$, and therefore $U=U^{**}=V^*$. Thus

$$\begin{aligned} U&=\{w: v'w\leq 1 \text{ for } v \in V\} \\ &=\left\{\alpha w: \max_{v \in V} v'w=1,\ 0\leq\alpha\leq 1\right\} \end{aligned}$$

(since V is a neighborhood of 0),

$$=\left\{\alpha w: \max_{x \in \partial R(t)} DT(x)e^{-At}w=1,\ 0\leq\alpha\leq 1\right\}.$$

Since every $u \in \partial U$ has $\lambda u \notin U$ for $\lambda>1$, we conclude our assertion.

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