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On the Existence and Uniqueness of *L*-integrable Solutions of a Certain Integral-Functional Equation

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It is well known that the solutions of even linear differential-functional equations of neutral type may have nasty discontinuities in their derivatives and this encourage us to consider the existence and uniqueness problems for such equations in setting like the Sobolev space $W^{1,1}$. On the other hand for certain "nice" neutral differential-functional equations one may lose a great deal of information by looking only for absolutely continuous solutions.

This is quite clear by the following example [4]

$$y'(t)=l\cdot y'(\beta t)+h(t), \qquad y(0)=0 \tag{$*$}$$

with $0\leqslant\beta\leqslant 1$, $l\in R^1$, $t\in[0,a]$, $h\in C^\infty[0,a]$, $h(0)=0$.

The result of paper [1] can be applied to this equation and we have the answer: if $|l/\beta|<1$, there is a unique absolutely continuous solution of equation $(*)$ with $L^1[0,a]$ derivative. By the results of papers [2], [3] we have the answer: if $|l\beta|<1$ there is a solution of equation $(*)$ with $C[0,a]$ derivative. On the other hand by the results of paper [4] we get the answer: if $n\geqslant 2$, $|l\beta^{n-1}|<1$ and $l\beta^j\neq 1$ for $j=0, 1,\cdots,n-2$ there exists a unique solution of equation $(*)$ in $C^n[0,a]$. However we note that, if $h(t)=t^p h_1(t)$, $h_1\in C[0,a]$ and $|l\beta^p|<1$ then by the result of [2], [3] there exists a solution of $(*)$ in $C^1[0,a]$ and it is unique in the class of functions such that $|y(t)|\leqslant Ct^{p+1}$, $t\in[0,a]$, $C=\text{const}$.

By this example we see that the sufficient conditions for existence of solution of equation $(*)$ depends on a class of functions in which this equation can be considered. The same observations relate to the nonlinear differential-functional equation of neutral type of the form

$$y'(t)=F(t,y(\alpha(t)),y'(\beta(t))), \qquad t\in[0,a] \tag{$**$}$$

with the initial condition $y(t)=\psi(t)$, $t\in(-\tau,0]$, $0\leqslant\tau\leqslant+\infty$. But, by the substitution $y'(t)=x(t)$ the initial problem $(**)$ reduces to the following one

$$x(t)=F\left(t,\psi(0)+\int_0^{\alpha(t)}x(s)ds,x(\beta(t))\right) \tag{$***$}$$

with $x(t)=\varphi(t)=\psi'(t)$, $t \in (-\tau, 0]$.

In view of this we shall consider a slightly more general integral-functional equation of the form

$$x(t)=F\Big(t, \int_0^{\alpha(t)} f(t, s, x(s))ds, x(\beta(t))\Big) \tag{1}$$

with the initial condition $x(t)=\varphi(t)$, $t \in (-\tau, 0]$.

The purpose of this paper is to give an unified approach to the existence and uniqueness problems for the equation (1) by which we arrive to the different sufficient conditions related to appropriate classes of functions in which this equation can be considered.

1. Preliminaries.

Let B denote a fixed Banach space with a norm $\|\cdot\|$ and $I=[0, a]$, $a \in R_+=[0, +\infty)$. By $L(D, B)$, D-an measurable subset of R^n, (R the set of real numbers), we shall denote the set of Bochner integrable functions defined on D with a range in B. Let $L^\infty(I, R_+)$ denotes the set of all measurable, essentially bounded and nonnegative functions defined on I.

We consider a function $F: I\times B\times B\to B$ such that $F(\cdot, u, v) \in L(I, B)$ for any $u, v \in B$ and a function $f: I\times I_\tau\times B\to B$ such that $f(\cdot, \cdot, u) \in L(I\times I_\tau, B)$, where $I_\tau=(-\tau, a]$, $\tau \in R_+$. We assume that $\alpha, \beta: I\to I_\tau$, $\alpha, \beta \in L(I, I_\tau)$ and $\alpha(t)\leqslant t$, $\beta(t)\leqslant t$ for $t \in I$.

Moreover we introduce:

Assumption A_1. The functions α, β have the property: if μ is the Lebesgue measure on R, then the inverse image by α and β of any subset of I of μ-measure equal to zero is μ-measurable.

Assumption A_2. There exists a bounded μ-measurable function $r: I\to[1, +\infty)$ such that for any μ-measurable subset $E \in [0, t]$, $t \in I$, the inequality

$$\mu E\leqslant r(t)\mu\beta(E), \qquad t \in I \tag{2}$$

holds. Further on we shall write "measurable" instead of "μ-measurable".

Assumption A_3. 1° There exist functions $k_1, k_2 \in L(I, R_+)$ and a nondecreasing function $l_1 \in L^\infty(I, R_+)$ such that

$$\|F(t, x, y)-F(t, \bar{x}, \bar{y})\|\leqslant k_1(t)\, \|x-\bar{x}\|+l_1(t)\, \|y-\bar{y}\| \tag{3}$$

for any $x, \bar{x}, y, \bar{y} \in B$, $t \in I$,

$$\|f(t, s, x)-f(t, s, y)\|\leqslant k_2(t)\, \|x-y\| \tag{4}$$

for any $(t, s) \in I \times I_\tau$, $x, y \in B$

2° A function $\varphi: (-\tau, 0] \rightarrow B$ is Bochner integrable.

Assumption A_4. Let $k \in L(I, R_+)$, $l, h \in L^\infty(I, R_+)$, $\bar{\alpha}, \bar{\beta} \in L(I, I)$, $\bar{\alpha}(t), \bar{\beta}(t) \in [0, t]$, $t \in I$, be given and let $\bar{\alpha}, \bar{\beta}$ satisfy Assumption A_1.

Let us define

$$\left.\begin{aligned} &\bar{\beta}_{i+1}(t) = \bar{\beta}(\bar{\beta}_i(t)),\ \bar{\beta}_0(t) = t, && t \in I \\ &l_{i+1}(t) = l(t) l_i(\bar{\beta}(t)),\ l_0(t) = 1, && t \in I \end{aligned}\right\} \tag{5}$$

and

$$(Lg)(t) = l(t) g(\bar{\beta}(t)),$$

$$(Kg)(t) = \int_0^t k(s) g(\bar{\alpha}(s)) ds,$$

$$(Mg)(t) = \sum_{i=0}^{\infty} l_i(t) g(\bar{\beta}_i(t))$$

for $g \in L^\infty(I, \boldsymbol{R}_+)$, with the almost everywhere convergence in I of the series.

It is obvious that

$$(L^i g)(t) = l_i(t) g(\bar{\beta}_i(t)) \quad \text{and} \quad Mg = \sum_{i=0}^{\infty} L^i g. \tag{6}$$

2. Some lemmas.

We have

Lemma 1. *If* 1° *Assumption A_4 is satisfied,*

2° $$Mh \in L^\infty(I, R_+),$$

3° $$M\tilde{k}/\tilde{k} \in L^\infty(I, R_+) \quad \textit{and} \quad \Lambda = \sup_I \text{ess} \frac{(M\tilde{k})(t)}{\tilde{k}(t)} < +\infty,$$

where

$$\tilde{k}(t) = \int_0^t k(s) ds,$$

then

a) *there exists $g_0 \in L^\infty(I, R_+)$ being a unique in $L^\infty(I, R_+)$ solution of the equation*

$$g = MKg + Mh \tag{7}$$

and

$$\lim_{n\to\infty} L^n g_0 = 0, \tag{8}$$

b) *the function g_0 is the unique in $M(I, R_+, g_0)$ solution of the equation*

$$g = Kg + Lg + h \tag{9}$$

where

$$M(I, R_+, g_0) = [g : g \in L^\infty(I, R_+), \|g\|_0 < +\infty]$$

and

$$\|g\|_0 = \inf_{c \in R_+} [c : g \leqslant c g_0],$$

c) *the function $g=0$ is in the class $M(I, R_+, g_0)$ the unique solution of the inequality*

$$g \leqslant Kg + Lg. \tag{10}$$

Proof. To prove a) we observe that $L^\infty(I, R_+)$ is a complete metric space with a metric defined by the norm

$$\|g\|_* = \sup_I \text{ess} \left(|g(t)| \exp\left(-\lambda \int_0^t k(s)ds \right) \right), \qquad \lambda \geqslant 0,$$

and that the operator MK is a contraction for $\lambda > \Lambda$. Indeed, if $g \in L^\infty(I, R_+)$ and $z = MKg$ then according to (6) we have

$$z(t) = \sum_{i=0}^{\infty} l_i(t) \int_0^{\beta_i(t)} k(\tau) \exp\left(\lambda \int_0^{\overline{\alpha}(t)} k(s)ds \right) g(\overline{\alpha}(\tau)) \exp\left(-\lambda \int_0^{\overline{\alpha}(\tau)} k(s)ds \right) d\tau$$
$$\leqslant \frac{\|g\|_*}{\lambda} \sum_{i=0}^{\infty} l_i(t) \left[\exp\left(\lambda \int_0^{\beta_i(t)} k(s)ds \right) - 1 \right].$$

However, by the inequality $\exp(\sigma t) - 1 \leqslant \sigma \exp t$ for $\sigma \in [0, 1]$, $t \geqslant 0$, we get

$$z(t) \leqslant \frac{\|g\|_*}{\lambda} \exp\left(\lambda \int_0^t k(s)ds \right) \sum_{i=0}^{\infty} l_i(t) \int_0^{\beta_i(t)} k(s)ds \left(\int_0^t k(s)ds \right)^{-1}$$

and

$$\|z\|_* \leqslant \frac{\Lambda}{\lambda} \|g\|_*, \qquad \frac{\Lambda}{\lambda} < 1.$$

The relation (8) follows by the equality

$$L^n g_0 = L^n MKg_0 + L^n Mh = \sum_{i=n}^{\infty} L^i Kg_0 + \sum_{i=n}^{\infty} L^i h$$

and by the convergence of the series

$$\sum_{i=0}^{\infty} L^i K g_0 \quad \text{and} \quad \sum_{i=0}^{\infty} L^i h.$$

Now we prove b). At first we observe that g_0 is the solution of (9). Indeed, by the equality $Mg=LMg+g$ we have

$$\begin{aligned} Kg_0+Lg_0+h &= Kg_0+L(MKg_0+Mh)+h \\ &= Kg_0+LMKg_0+LMh+h=MKg_0+Mh=g_0. \end{aligned}$$

Now if $\tilde{g} \in M(I, R_+, g_0)$ is a solution of (9) then by induction we get

$$\tilde{g}=\sum_{i=0}^{n-1} L^i h+\sum_{i=0}^{n-1} L^i K\tilde{g}+L^n\tilde{g}.$$

Because there exists $c \in R_+$ such that $0 \leqslant \tilde{g} \leqslant cg_0$ then by (8) we infer $L^n\tilde{g} \to 0$ for $n \to \infty$. As a consequence of this we find that $\tilde{g}$ satisfies (7). In view of the unicity proved for this equation we conclude $g=g_0$.

Finally we prove c). If $g \in M(I, R_+, g_0)$ satisfies (10) then we have $L^n g \to 0$ for $n \to \infty$ and

$$g \leqslant \sum_{i=0}^{n-1} L^i K g+L^n g.$$

Letting $n \to \infty$ we get $g \leqslant MKg$ and by the contraction property of the operator MK we find that $g=0$. Lemma is proved.

Remark 1. If for some $t^* \in I$ $\tilde{k}(t^*)=0$, then $(M\tilde{k})(t^*)=0$ and we can take $(M\tilde{k})(t^*)/\tilde{k}(t^*)=0$.

We note that by Lemma 1 the existence and uniqueness assertion concerning equation (9) (being the basic fact for our purpose) is related to the "individual" properties 2° and 3° of the given functions h, k, l and $\bar{\beta}$.

Let us now indicate some conditions being sufficient in order the assumptions 2° and 3° of Lemma 1 to be fulfilled.

(i) For any $k \in L(I, R_+)$, $h \in L^\infty(I, R_+)$ and $l(t) \equiv \bar{l} \in R_+$ 2° and 3° obviously are both fulfilled if $\bar{l}<1$. However we have

(ii) If $h, k \in L^\infty(I, R_+)$, $l(t) \equiv \bar{l} \in R_+$, $\bar{\beta}(t) \leqslant \beta_* \cdot t$, $\beta_* \in [0, 1]$, then 2° and 3° are both fulfilled if the series

$$\sum_{i=0}^{\infty} \bar{l}^i h(\beta_i(t))$$

is convergent to a function from $L^\infty(I, R_+)$ and $\bar{l}\beta_*<1$. The last condition $\bar{l}\beta_*<1$ is sufficient for 2° and 3° if we suppose that $h(t) \leqslant \bar{h}t$ for some $\bar{h} \in R_+$.

(iii) If $k(t) \leqslant k_* \cdot t^q$ for some $k_* \in R_+$ and $q \in (-1, +\infty)$, $l(t)=\bar{l} \in R_+$, $\bar{\beta}(t) \leqslant \beta_* \cdot t$ and if $\bar{l} \cdot \max(\beta_*^p, \beta_*^{q+1})<1$, $h(t) \leqslant \bar{h}t^p$, $p \in R_+$, then 2° and 3° hold.

(iv) If $h, k \in L^\infty(I, R_+)$, $\bar\beta(t) \leqslant \beta_* \cdot t$, $\beta_* \in [0, 1)$ and $l(t) \leqslant \bar l t^s$, $s > 0$, then 2° and 3° hold.

We have also

Lemma 2. *If* 1° *Assumption* A_4 *is satisfied,*

2° $$\Lambda^* = \sup_I \text{ess} \left[t^{-p} \sum_{i=0}^{\infty} l_i(t)(\bar\beta_i(t))^p \right] < +\infty$$

for some $p \in R_+$,

3° $$Mh \in V_p(I, R_+),$$

where

$$V_p(I, R_+) = [g : g \in L^\infty(I, R_+), \|g\|_{**} < +\infty]$$

and

$$\|g\|_{**} = \sup_I \text{ess} \left(t^{-p} |g(t)| \exp\left(-\lambda \int_0^t k(s)ds \right) \right), \qquad \lambda > \Lambda^*,$$

then the assertion of Lemma 1 *hold if the classes* $L^\infty(I, R_+)$ *and* $M(I, R_+, g_0)$ *are both replaced by* $V_p(I, R_+)$.

Proof. First we prove a). It is obvious that $V_p(I, R_+)$ is a complete metric space with a metric defined by the norm $\|\cdot\|_{**}$. Now the operator MK is a contraction in $V_p(I, R_+)$. Indeed, if $g \in V_p(I, R_+)$ and $z = MKg$ we find that $z + Mh \in V_p(I, R_+)$ and

$$\begin{aligned} z(t) &= \sum_{i=0}^{\infty} l_i(t) \int_0^{\beta_i(t)} k(\tau) \exp\left(\lambda \int_0^{\bar\alpha(\tau)} k(s)ds \right) \\ &\quad \times (\bar\alpha(\tau))^p \left[(\bar\alpha(\tau))^{-p} g(\bar\alpha(\tau)) \exp\left(-\lambda \int_0^{\bar\alpha(t)} k(s)ds \right) \right] d\tau \\ &\leqslant \frac{\|g\|_{**}}{\lambda} \sum_{i=0}^{\infty} l_i(t)(\bar\beta_i(t))^p \left[\exp\left(\lambda \int_0^{\bar\beta_i(t)} k(s)ds \right) - 1 \right] \\ &\leqslant \frac{\|g\|_{**}}{\lambda} \sum_{i=0}^{\infty} l_i(t)(\bar\beta_i(t))^p \left[\exp\left(\lambda \int_0^t k(s)ds \right) \int_0^{\bar\beta_i(t)} k(s)ds \left(\int_0^t k(s)ds \right)^{-1} \right] \end{aligned}$$

whence we obtain

$$\|z\|_{**} \leqslant \frac{\Lambda_*}{\lambda} \|g\|_{**}, \qquad \frac{\Lambda^*}{\lambda} < 1.$$

The remaining argument is the same as in the proof of Lemma 1.

Remark 2. It is clear that in Lemma 2 the case $a=+\infty$ is not excluded and now $L(I, R_+)$, $L^\infty(I, R_+)$ can be replaced by $L_{\mathrm{loc}}(I, R_+)$ and $L^\infty_{\mathrm{loc}}(I, R_+)$.

More effective is the following

Lemma 3. *If* 1° *Assumption* A_4 *is satisfied,*

2° $h(t)\leqslant \bar{h}t^p$ *for some* $\bar{h}, p \in R_+$,

3° $l(t)\leqslant \bar{l} \in R_+$, $\bar{\beta}(t)\leqslant \beta_* t$, $\beta_* \in [0, 1]$, $t \in I$,

4° $\bar{l}\beta_*{}^p<1$,

then the assertion of Lemma 1 *holds with* $L^\infty(I, R_+)$, $M(I, R_+, g_0)$ *both replaced by* $V_p(I, R_+)$.

Proof. This lemma is implied by the previous one.

Remark 3. Under the assumptions of Lemma 3 it is easy to prove directly that the operator $K+L$ is a contraction in $V_p(I, R_+)$. In fact, using the norm $\|\cdot\|_{**}$ we find for $z=Kg+Lg$, $g \in V_p(I, R_+)$

$$\|z\|_{**}\leqslant\left(\frac{1}{\lambda}+\bar{l}\beta_*^p\right)\|g\|_{**}.$$

3. Existence and uniqueness.

Now we go back to the equation (1). We investigate this equation under assumptions mentioned at the begining of this paper. To solve equation (1) we take a function $x_0 \in L(I_\tau, B)$ such that $x_0(t)=\varphi(t)$ for $t \in (-\tau, 0]$ and we define the sequence $\{x_n\}$

$$x_{n+1}(t)=F\left(t, \int_0^{\alpha(t)} f(t, s, x_n(s))ds, x_n(\beta(t))\right), \qquad t \in I \tag{11}$$

and $x_{n+1}(t)=\varphi(t)$ for $t \in (-\tau, 0]$. We shall prove that $\{x_n\}$ is convergent (in $L(I_\tau, B)$) to a solution of equation (1).

Put

$$u_n(t)=\|x_n(t)-x_0(t)\|, \quad v_n(t)=\int_0^t u_n(s)ds, \quad h(t)=v_1(t) \tag{12}$$

for $t \in I$ and $n=0, 1, \cdots$.

$$\begin{aligned} k(t)=k_1(t)k_2(t), \quad \bar{\alpha}(t)=\max\,[0, \alpha(t)], \quad \bar{\beta}(t)=\sup_{0\leqslant s\leqslant t}\max\,[0, \beta(s)] \\ l(t)=l_1(t)r(t) \qquad t \in I. \end{aligned} \tag{13}$$

We have

Lemma 4. *If* 1° *Assumptions* A_1–A_3 *are satisfied,* 2° *the functions* $h, k, l, \overline{\alpha}, \overline{\beta}$ *defined by* (12) *and* (13) *fulfil the assumptions of Lemma* 1, *then*

$$\int_0^t \|x_n(s)-x_0(s)\|\,ds = v_n(t) \leqslant g_0(t), \qquad t \in I,\ n=0, 1, \cdots \tag{14}$$

where g_0 *is the function defined in Lemma* 1.

Proof. Using Assumption A_3 we find for $t \in I$

$$\begin{aligned} u_{n+1}(t) &= \|x_n(t)-x_0(t)\| \\ &\leqslant \left\| F\Big(t, \int_0^{\alpha(t)} f(t, s, x_n(s))ds,\ x_n(\beta(t))\Big) - F\Big(t, \int_0^{\alpha(t)} f(t, s, x_0(s))ds,\ x_0(\beta(t))\Big)\right\| \\ &\quad + u_1(t) \leqslant k(t) \int_0^{\alpha(t)} u_n(s)ds + l_1(t)u_n(\beta(t)) + u_1(t). \end{aligned}$$

Integrating both sides of this inequality on the interval $[0, t]$ and using the monotonicity of l_1 we obtain

$$v_{n+1}(t) \leqslant \int_0^t k(\tau)v_n(\overline{\alpha}(\tau))d\tau + l_1(t)\int_0^t u_n(\beta(\tau))d\tau + h(t).$$

By Assumption A_2 and by the same argument as in the paper [1] we find the inequality

$$\int_0^t u_n(\beta(\tau))d\tau \leqslant r(t) \int_0^{\overline{\beta}(t)} u_n(s)ds.$$

In view of this we get

$$v_{n+1}(t) \leqslant \int_0^t k(s)v_n(\overline{\alpha}(s))ds + l(t)v_n(\overline{\beta}(t)) + h(t) \qquad \text{for } t \in I,\ n=0, 1, \cdots. \tag{15}$$

We have obviously $v_1(t) = h(t) \leqslant g_0(t)$, $t \in I$, where g_0 is defined in Lemma 1. By induction from (15) we arrive to the assertion of lemma.

In order to prove the convergence of the sequence $\{x_n\}$ we define the following one

$$g_{n+1} = Kg_n + Lg_n, \qquad n=0, 1, \cdots \tag{16}$$

where g_0 is from Lemma 1.

First of all we state

Lemma 5. *If the assumptions of Lemma* 1 *are satisfied and* $\{g_n\}$ *is defined by* (16), *then*

$$0 \leqslant g_{n+1} \leqslant g_n \leqslant g_0 \qquad \text{for } n=0, 1, \cdots$$

and

$$\lim_{n\to\infty} g_n = 0 \qquad \text{a.e. } in\ I.$$

The proof of this lemma is obvious.

Put

$$u_{n,p}(t) = \|x_{n+p}(t) - x_n(t)\|, \qquad t \in I,\ n, p = 0, 1, \cdots.$$

Lemma 6. *If the assumptions of Lemma* 4 *are satisfied and* $y \in L(I_\tau, B)$, $y(t) = \varphi(t)$, $t \in (-\tau, 0]$ *is a solution of equation* (1) *satisfying the condition*

$$\int_0^t \|y(s) - x_0(s)\|\, ds \leqslant g_0(t), \qquad t \in I, \tag{17}$$

then

$$v_{n,p}(t) = \int_0^t \|x_{n+p}(s) - x_n(s)\|\, ds \leqslant g_n(t) \tag{18}$$

and

$$\int_0^t \|y(s) - x_n(s)\|\, ds \leqslant g_n(t) \tag{19}$$

for $t \in I$, $n, p = 0, 1, \cdots$.

Proof. By Lemma 4 we see that (18) hold for $n=0$ and $p=0, 1, \cdots$. If we suppose (18) then by assumptions introduced we get

$$u_{n+1,p}(t) \leqslant k(t) \int_0^{\alpha(t)} u_{n,p}(s) ds + l_1(t) u_{n,p}(\beta(t)),$$

and by the same argument as in the proof of Lemma 4 we find

$$\begin{aligned} v_{n+1,p}(t) &\leqslant \int_0^t k(s) v_{n,p}(\bar\alpha(s)) ds + l(t) v_{n,p}(\bar\beta(t)) \\ &\leqslant \int_0^t k(s) g_n(\bar\alpha(s)) ds + l(t) g_n(\bar\beta(t)) = g_{n+1}(t). \end{aligned}$$

Now (18) follows by induction. In the same way using (17) and (16) we prove the relation (19). Thus the lemma is proved.

Put

$$L(I_\tau, B, g_0) = [y : y \in L(I_\tau, B),\ y(t) = \varphi(t),\ t \in (-\tau, 0],\ \|v_y\|_0 < +\infty],$$

where

$$v_y(t)=\int_0^t \|y(s)-x_0(s)\|\,ds, \qquad t \in I.$$

Theorem 1. *If* 1° *Assumptions* A_1–A_3 *are satisfied,*

2° *the functions* $h, k, l, \bar{\alpha}, \bar{\beta}$ *defined by* (2), (12), (13) *satisfy Assumption* A_4,

3° $Mh, M\tilde{k}/\tilde{k} \in L^\infty(I, R_+)$, *where*

$$\tilde{k}(t)=\int_0^t k(s)ds,$$

then there exists a solution $\bar{x} \in L(I_\tau, B)$ *of equation* (1), *this solution is unique in the class* $L(I_\tau, B, g_0)$ *and*

$$\int_0^t \|\bar{x}(s)-x_n(s)\|\,ds \leqslant g_n(t), \qquad t \in I. \tag{20}$$

Proof. By Lemma 6—evaluation (18)—we infer convergence of the sequence $\{x_n\}$ to some $\bar{x} \in L(I_\tau, B)$. Letting $p \to \infty$ in (18) we have

$$\int_0^t \|\bar{x}(s)-x_n(s)\|\,ds \leqslant g_n(t), \qquad t \in I,\ n=0, 1, \cdots.$$

Now we check that $\bar{x}$ is a solution of equation (1). We have

$$\begin{aligned}\Big\|\bar{x}(t)-F\Big(t, \int_0^{\alpha(t)} f(t, s, \bar{x}(s))ds, \bar{x}(\beta(t))\Big)\Big\| &\leqslant \|\bar{x}(t)-x_{n+1}(t)\| \\ &+\Big\|F\Big(t, \int_0^{\alpha(t)} f(t, s, x_n(s))ds, x_n(\beta(t))\Big)-F\Big(t, \int_0^{\alpha(t)} f(t, s, \bar{x}(s))ds, \bar{x}(\beta(t))\Big)\Big\| \\ &\leqslant \|\bar{x}(t)-x_{n+1}(t)\|+k(t)\int_0^{\alpha(t)} \|x_n(s)-\bar{x}(s)\|\,ds+l_1(t)\,\|x_n(\beta(t))-\bar{x}(\beta(t))\|.\end{aligned}$$

After integrating of this inequality we get

$$\begin{aligned}\int_0^t \Big\|\bar{x}(\tau)-F\Big(\tau, \int_0^{\alpha(\tau)} f(\tau, s, x(s))ds, \bar{x}(\beta(\tau))\Big)\Big\|\,d\tau &\leqslant \int_0^t k(s)g_n(\bar{\alpha}(s))ds \\ &+l(t)g_n(\bar{\beta}(t))+g_{n+1}(t) \leqslant 2g_{n+1}(t).\end{aligned}$$

Now by letting $n \to \infty$ we find the assertion.

To prove the uniqueness let us suppose that $y \in L(I_\tau, B, g_0)$ is a solution of equation (1). We have then for some $c \in R_+$

$$\int_0^t \|y(s)-x_0(s)\|\,ds \leqslant cg_0(t), \qquad t \in I.$$

By the same argument as in proving of (19) we find

$$\int_0^t \|y(s)-x_n(s)\|\, ds \leqslant c g_n(t), \qquad t \in I,\ n=0, 1, \cdots$$

and as a consequence of this $y(t)=\bar{x}(t)$ in $L(I_\tau, B)$. Thus the theorem is proved.

Combinning Assumptions A_1–A_4 with one from the conditions (i), (ii), (iii), (iv) we find another existence theorem for equation (1).

We have then

Theorem 2. *If assumptions* $1°, 2°$ *of Theorem* 1 *are fulfilled and* $\sup_I l_1(t)r(t) =\bar{l}<1$, *then the assertion of Theorem* 1 *hold.*

Remark 4. We note, if $l_1(t)=\bar{l}_1 \in R_+$ and $\beta(t)=\beta_* \cdot t$, $0 \leqslant \beta_* \leqslant 1$, then $r(t) \equiv 1/\beta_*$ and we have the condition $\bar{l}_1/\beta_*<1$ mentioned at the begining of the paper.

But if we have more information on a given functions we find:

Theorem 3. *If assumptions* $1°, 2°$ *of Theorem* 1 *are satisfied and* (iii) *holds then the assertion of Theorem* 1 *holds.*

Remark 5. Because

$$h(t)=\int_0^t \left\| F\left(\tau, \int_0^{\alpha(\tau)} f(\tau, s, x_0(s))ds, x_0(\beta(\tau))\right)-x_0(\tau)\right\| d\tau$$

then the condition $h(t) \leqslant \bar{h}t^p$ assumed in (iii) depends on x_0. If the integrand in the definition of h is bounded then we have $h(t) \leqslant \bar{h}t$. Now if $q=0$, $l_1(t) \equiv \bar{l}_1 \in R_+$ and $\beta(t)=\beta_* \cdot t$, $\beta_* \in [0, 1]$, then we get the sufficient condition: $\bar{l}_1<1$.

Using Lemma 3 we obtain the better result

Theorem 4. *If assumptions* $1°, 2°$ *of Theorem* 1 *are satisfied and*

1° $h(t) \leqslant \bar{h}t^p$ *for some* $\bar{h}, p \in R_+$,

2° $\sup_I l_1(t)r(t) \leqslant \bar{l} \in R_+$,

3° $\bar{\beta}(t) \leqslant \beta_* \cdot t$, $\beta_* \in [0, 1]$, $t \in I$,

4° $\bar{l}\beta_*^p<1$,

then there exists in $V_p(I, R_+)$ *a unique solution of equation* (1).

Remark 6. If under assumptions of Theorem 4 we have $\beta(t)=\beta_* \cdot t$ and $l_1(t) \equiv \bar{l}_1 \in R_+$, then the condition 4° in this theorem reads $\bar{l}_1\beta_*^{p-1}<1$.

This result close to the result obtained in [2], [3] where equation (1) was considered in the space $C(I, B)$. It is important to note that not all results established in [1] can be derived from the results of the present paper. For instance, unfortunately in this paper we can not take $\beta(t)=t^2$ for $t \in [0, 1/2]$, in this case the function r defined by (2) is unbounded. It seems that some result can be obtained only if (1) is considered in $C(I, B)$ but not in $L(I, B)$.

Finally we note that the method presented in this paper can be used in order to obtain the existence and uniqueness result for more general integral-functional equations for instance of the form

$$x(t)=F\left(t, \int_0^{\alpha_1(t)} f_1(t, s, x(s))ds, \cdots, \int_0^{\alpha_n(t)} f_n(t, s, x(s))ds, x(\beta_1(t)), \cdots, x(\beta_m(t))\right).$$

In this case we shall meet only some technical complications.

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