On a Linear System of Pfaffian Equations with Regular Singular Points

By

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Introduction.

The purpose of this paper is to extend some well known theorems on ordinary differential equations to Pfaffian systems.

We consider a completely integrable system

\begin{equation}
\frac{dX}{dt} = \left( \sum_{i=1}^{m} \frac{P_i(x)}{x_i} dt \right) X
\end{equation}

where $X$ is an unknown $m \times m$ matrix and $P_i(x)$ is an $m \times m$ matrix holomorphic at $x=0$, i.e. $P_i(x)$ permits a power series expansion

$$P_i(x) = \sum_{k \geq 0} P_{i,k} x^k,$$

convergent in a neighborhood of $x=0$. Here $k$ denotes a multi-index $(k_1, \cdots, k_n)$, $k_i$ nonnegative integer, $0=(0, \cdots, 0)$ and $x^k = x_1^{k_1} \cdots x_n^{k_n}$. For two multi-indices $k$ and $l$, "$k \geq l$" means "$k_i \geq l_i$ for all $i$" and "$k > l$" means "$k \geq l$ and $k_i > l_i$ for some $i$". We propose to construct the solutions of (1) concretely and to find out the dominant coefficients of \{\$P_{i,k}$\} which determine the local behavior of the solutions.

As in the local theory of ordinary differential equations, our study is divided into two parts:

(i) to find out a formal transformation which reduces (1) to a solvable equation,

(ii) to prove the convergence of the formal transformation. We call the former the formal part and the latter the analytic part. The analytic part for system (1) is easily achieved; in fact, the technique is the same as in that for ordinary differential equations. However, the formal part for system (1) contains an essential difficulty, which does not appear in that for ordinary differential equations, because the recursion formulas to determine the coefficients of a formal transfor-
tion and the integrability condition are extremely complicated for our case. Thus our effort is concentrated on making clear the effect of a transformation on the consequent equation and the role of integrability condition. We show that the complication can be resolved by decomposing a transformation into a sequence of simple transformations of special form. As far as the authors know, the idea of decomposing a transformation in the formal theory is due to Professor M. Hukuhara.

In Section 1, we prepare, in advance, a convergence theorem guaranteeing that the formal transformations considered in this paper are actually convergent. In Section 2, we state the integrability condition in terms of the coefficients \( \{P_{\ell,i}\} \), since the expression of the integrability condition in this form is required to achieve the calculations below. In Section 3, first we give the definition of the notion that a system of the form (1) has a 'reduced' form. Next we prove that by a suitable change of variables \( X = U(x)Y \) with \( U(x) \) invertible and holomorphic at \( x = 0 \), system (1) is transformed to a system

\[
(2) \quad dY = \left( \sum_{i=1}^{n} \frac{Q_i(x)}{x_i} \right) Y
\]

which has a 'reduced' form. We determine \( U(x) \) as a product of infinitely many simple matrices of special form. This section is the main part of our paper. Indeed, once the reduction to a 'reduced' form is completed, there remains no difficulty. In Section 4, we show that a substitution

\[
Y = x_1^{L_1} \cdots x_n^{L_n} Z
\]

with \( L_i = \text{diag} (l_i^1, \ldots, l_i^n) \), \( l_i^j \) being a nonnegative integer, changes system (2), which has a 'reduced' form, to a system

\[
(3) \quad dZ = \left( \sum_{i=1}^{n} \frac{B_i}{x_i} dx_i \right) Z
\]

with constant matrices \( B_1, \ldots, B_n \). Combining the results in Sections 3 and 4, we can establish one of our two main theorems, which is stated in the last section.

In the arguments above, it was not the question if some \( P_i(x)/x_i \) is holomorphic at \( x = 0 \) or not. However, in the first alternative, we had better eliminate the variable \( x_i \), before making the change of variables studied in Sections 3 and 4. Thus Section 5 is devoted to this elimination problem; in more detail, it shows that if \( P_i(x)/x_i \) is holomorphic at \( x = 0 \) for \( \nu < i \leq n \), system (1) can be changed to the system

\[
(4) \quad dY = \left( \sum_{i=1}^{n} \frac{P_i(x, \cdots, x_i, 0, \cdots, 0)}{x_i} dx_i \right) Y
\]
by a suitable substitution $X = U(x)Y$ with $U(x)$ invertible and holomorphic at $x = 0$. Combining the results of Sections 3, 4 and 5, we have the other main theorem, which is also stated in the last section.

§ 1. Convergence theorem.

In this section, as was announced in the introduction, we shall prove the following convergence theorem. It is an extension of a well known theorem for one variable (cf. [4]) to several variables.

**Theorem 1.** Let

\[
\text{(5) } du = \left( \sum_{i=1}^{n} \frac{F_i(x)}{x_i} \, dx_i \right) u
\]

be a completely integrable system, where $u$ is a vector and

\[
F_i(x) = \sum_{k \geq 0} F_{i,k} x^k, \quad i = 1, \ldots, n
\]

is a matrix convergent and holomorphic for $|x| < \varepsilon$. Then any formal power series solution of (5) written as

\[
\text{(6) } u(x) = \sum_{k \geq 0} u_k x^k
\]

converges for $|x| < \varepsilon$ and represents a holomorphic solution of (5).

**Proof.** Since (6) formally satisfies (5), we have

\[
F_{i,0} u_0 = 0
\]

and

\[
(F_{i,0} - k_i I) u_k = - \sum_{0 \leq l \leq k} F_{i,l} u_{k-l}
\]

for all $i = 1, \ldots, n$. There exists a positive integer $k_i^0$ such that $\det (F_{i,0} - k_i I) \neq 0$ and $\| (F_{i,0} - k_i I)^{-1} \| \leq 1$ for all $k_i > k_i^0$. Here $\| \cdot \|$ denotes the usual Euclidian norm. Hence

\[
\text{(7) } \| u_k \| \leq \sum_{0 \leq l \leq k} f_l \| u_{k-l} \|
\]

for $k \leq k^0$, where $f_l = \max_{1 \leq i \leq n} \| F_{i,l} \|$. We note that the series $f(x) = \sum f_i x^i$ converges and so is holomorphic for $|x| < \varepsilon$. Now we introduce the scalar majorizing function $\nu(x)$ defined by
\[ (8) \quad v(x) = f(x) v(x) + \|u_0\| + \sum_{0 < k \leq k^0} \left( \sum_{0 < i \leq k} f_i \|u_{k-i}\| \right) x^k. \]

Since \( f(0) = 0 \), there exists a positive number \( \epsilon' < \epsilon \) such that \( v(x) \) is holomorphic for \( |x| < \epsilon' \). Let
\[ (9) \quad v(x) = \sum_{k \geq 0} v_k x^k, \quad |x| < \epsilon'. \]

Substitution of (9) in (8) leads to the recursion formulas
\[ v_k = \|u_k\|, \quad k \leq k^0 \]
and
\[ v_k = \sum_{0 < i \leq k} f_i \|u_{k-i}\|, \quad k > k^0. \]

Comparison with (7) shows \( \|u_k\| \leq v_k \) for all \( k \). Hence the convergence of the series (9) implies the convergence of (6) in \( |x| < \epsilon' \). The completion of the proof is now immediate.

\section*{§ 2. Integrability condition.}

The integrability condition for a system \( dX = \Omega(x)X \) is
\[ (10) \quad d\Omega(x) - \Omega(x) \wedge \Omega(x) = 0. \]

In case \( \Omega(x) = \sum_{i=1}^{n} \frac{P_i(x_i)}{x_i} dx_i \) where
\[ P_i(x) = \sum_{k \geq 0} P_{i,k} x^k \]
is convergent and holomorphic in a neighborhood of \( x=0 \), the condition (10) is equivalent to
\[ (11)_{i,j;k} \quad k_j P_{i,k} - k_i P_{j,k} + \sum_{k'+k''=k} [P_{i,k'}, P_{j,k''}] = 0 \]
for all \( i, j = 1, \ldots, n \) and \( k = (k_1, \ldots, k_n) \). Here \([,]\) denotes the usual bracket of matrices: \([A, B] = AB - BA\).

\section*{§ 3. Reduction to a ‘reduced’ form.}

The purpose of this section is to reduce system (1) to a system which has a form as simple as possible by a change of variables \( X = U(x)Y \) with \( U(x) \) invertible and holomorphic at \( x=0 \). We shall begin with giving the following definition.

\textbf{Definition.} We say the equation
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\[ dX = \left( \sum_{i=1}^{n} \frac{P_i(x)}{x_i} dx_i \right) X \]

is ‘reduced’ with respect to \((k_i, (\alpha, \beta))\) if \(a_i^\alpha - a_i^\beta - k_i \neq 0\) for some \(i\) implies \(p_{jk}^\alpha = 0\) for all \(j=1, 2, \ldots, n\), where \(k = (k_1, \ldots, k_n)\), \(P_i(x) = \sum_{k \geq 0} P_{i,k} x^k\), \(P_{i,k} = (p^{\alpha}_{ik} \cdots p^{\beta}_{ik})\) and \(P_{i,0} = (a_{i}^{\alpha} \cdot \cdots \cdot a_{i}^{\beta})\). Furthermore, we say the equation has a ‘reduced’ form if it is reduced with respect to all \((k, (\alpha, \beta))\).

First we shall show that by a suitable formal transformation \(X = U(x)Y\), \(U(x) = \sum_{k \geq 0} U_k x^k\) being a formal power series of \(x\), (1) can be changed to a system which has a ‘reduced’ form. Next we shall prove the convergence of the formal power series \(\sum_{k \geq 0} U_k x^k\). We remark that the formal reduction will be carried out by decomposing a transformation \(X = \left( \sum_{k \geq 0} U_k x^k \right) Y\) into a product of infinitely many transformations.

### 3.1. Formal reduction

Before exposing the process to decompose \(X = \left( \sum_{k \geq 0} U_k x^k \right) Y\), we explain the reason why such a decomposition is necessary. Let (2) be the system changed from (1) by \(X = U(x)Y\), then we have

\[ Q_i(x) = U(x)^{-1} P_i(x) U(x) - U(x)^{-1} x_i \frac{\partial U(x)}{\partial x_i}, \quad i = 1, \ldots, n. \]

In the usual way, we substitute \(U(x) = \sum_{k \geq 0} U_k x^k\) in the above equations and get infinitely many relations between the coefficients \(\{U_k\}, \{P_{i,k}\}, \{Q_{i,k}\}\) (the coefficients in \(Q_i(x) = \sum_{k \geq 0} Q_{i,k} x^k\)), and determine the values of \(\{U_k\}\) such that almost all of \(\{Q_{i,k}\}\) vanish. Considering these relations as recursion formulas to determine \(\{U_k\}\) successively, we can easily verify that each \(U_k\) must satisfy \(n\) relations at the same time. The integrability condition must assure that \(U_k\) can be determined compatibly. However, the recursion formulas and the integrability condition are so complicated that it is very difficult to see to what extent system (1) can be simplified and that even if a canonical form is incidentally found out, it is hard to show that \(U_k\) can be determined compatibly.

Thus, in order to make clear the effect of the coefficients \(\{U_k\}\) on the consequent equation and the role of integrability condition, we decompose the transformation \(X = \left( \sum_{k \geq 0} U_k x^k \right) Y\) into a product of simple transformations.

Now we return to our subject. It is easy to see that every formal power series \(U(x) = \sum_{k \geq 0} U_k x^k\) (det \(U_0 \neq 0\)) can be uniquely decomposed as follows
\[ U(x) = U_0 \cdot U_1(x) \cdots U_N(x) \cdots \]

where \( U_0 \) is the constant matrix of the given \( \sum_{k \geq 0} U_k x^k \) and

\[ U_N(x) = U_N^{(1,m)}(x) \cdot U_N^{(2,m)}(x) \cdots U_N^{(m,m)}(x) \times U_N^{(1,m-1)}(x) \cdot U_N^{(2,m-1)}(x) \cdots U_N^{(m,m-1)}(x) \]

\[ \vdots \]

\[ \times U_N^{(1,1)}(x) \cdot U_N^{(2,1)}(x) \cdots U_N^{(m,1)}(x), \]

\[ U_N^{(g,\delta)}(x) = I + \sum_{|k|=N} U_{N,k}^{(g,\delta)} x^k. \]

Here \( |k| = \sum_{i=1}^{n} k_i \), and \( U_N^{(g,\delta)} \) is a constant matrix of which the \((\gamma, \delta)\) component is zero except for \((\gamma, \delta) = (\alpha, \beta)\). (The \((\alpha, \beta)\) component of \( U_N^{(g,\delta)} \) will be denoted by \( u_{N,k}^{g,\delta} \).) Conversely, the infinite product of \( \{U_N^{(g,\delta)}(x)\} \) of the special form as above can be expanded into a formal power series of \( x \). In other words, the formal transformation \( X = \left( \sum_{k \geq 0} U_k x^k \right) Y \) can be uniquely decomposed into a sequence of transformations \( \{X = U_N^{(g,\delta)}(x)Y\} \) and conversely a sequence of transformations \( \{X = U_N^{(g,\delta)}(x)Y\} \) determines a formal transformation \( \sum_{k \geq 0} U_k x^k \).

If we decompose \( X = U(x)Y \) in this way, then we can easily see the effect of each \( U_N^{(g,\delta)}(x) \) on the consequent equation and the role of the integrability condition. We shall show how to determine \( U_0, U_1^{(1,m)}(x), \ldots, U_k^{(m,m)}(x), \ldots, U_N^{(g,\delta)}(x), \ldots \) successively.

**Determination of \( U_0 \).** By the integrability condition \((11)_{\gamma,\delta,0}\), the constant matrices \( P_{\gamma,0} \) in \( P_\gamma(x) = \sum_{k \geq 0} P_{\gamma,k} x^k \) and \( P_{\delta,0} \) in \( P_\delta(x) = \sum_{k \geq 0} P_{\delta,k} x^k \) in \((1)\) are commutative: \([P_{\gamma,0}, P_{\delta,0}] = 0\) for all \( i, j = 1, \ldots, n \). Therefore, by the elementary matrix theory, we can choose a nonsingular constant matrix \( U_0 \) such that

\[ A_\iota U_0 = P_{\iota,0} U_0, \ i = 1, \ldots, n \text{ is lower triangular}, \]

\[ \text{(i)} \]

\[ \text{if } a^{\alpha i} - a^{\beta i} \neq 0 \text{ for some } i, \text{ then } a^{\iota \beta} = 0 \text{ for all } j = 1, \ldots, n, \text{ where } A_\iota = (a^{\iota \beta}). \]

We note that the equation transformed by \( X = U_\gamma Y \) is ‘reduced’ with respect to \((0, (\gamma, \delta))\) for all \((\gamma, \delta)\).

**Determination of \( U_1^{(1,m)}(x), U_1^{(2,m)}(x), \ldots, U_N^{(g,\delta)}(x), \ldots \).** The coefficient \( u_{N,k}^{g,\delta}(|k| = N) \) is determined successively by the induction process described below. We apply the following proposition to determine \( U_N^{(g,\delta)}(x) \), regarding as the equation in \( X \) in the proposition the equation transformed from \((1)\) by \( X = \)
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\[ U_0 \cdot U_1^{(1,m)}(x) \cdot \cdots \cdot U_N^{(\alpha-1,\beta)}(x) Y \]
which is obviously completely integrable. By introducing the notation \((\gamma, \delta) < (\alpha, \beta)\) for \(\delta > \beta\) or \(\delta = \beta\), \(\gamma < \alpha\), the induction process is described as follows.

**Proposition 1** (Induction process). Assume that the completely integrable system

\[
dX = \left( \sum_{i=1}^{n} \frac{P_i(x)}{x_i} dx_i \right) X
\]

with

\[
P_i(x) = \sum_{k \geq 0} P_{i,k} x^k, \quad P_{i,k} = (p_{ik}^{**})
\]
is 'reduced' with respect to \((k, (\gamma, \delta))\) both for the cases when \(|k| < N\), \((\gamma, \delta)\) arbitrary and when \(|k| = N\), \((\gamma, \delta) < (\alpha, \beta)\). Assume further \(P_{i,0} = A_i = (a_{i,k}^{**})\) satisfies the condition (12). Then one finds a transformation

\[
X = U_N^{(a, \gamma)}(x) Y
\]
such that the consequent equation

\[
dY = \left( \sum_{i=1}^{n} \frac{Q_i(x)}{x_i} dx_i \right) Y
\]

with

\[
Q_i(x) = \sum_{k \geq 0} Q_{i,k} x^k, \quad Q_{i,k} = (q_{ik}^{**})
\]
is 'reduced' with respect to \((k, (\gamma, \delta))\) both for the cases when \(|k| < N\), \((\gamma, \delta)\) arbitrary and when \(|k| = N\), \((\gamma, \delta) \leq (\alpha, \beta)\).

Furthermore

(13) \[ Q_{i,k} = P_{i,k} \quad \text{for} \quad k \mid |k| < N, \quad i = 1, \ldots, n, \]
(14) \[ q_{ik}^{\beta} = p_{ik}^{\beta} \quad \text{for} \quad k \mid |k| = N, \quad (\gamma, \delta) < (\alpha, \beta), \quad i = 1, \ldots, n. \]

If \(a_{i_0}^{(\alpha} - a_{i_0}^{(\beta} - k_{i_0} \neq 0\) for some \(i_0\), then the value of \(u_{N,k}^{(\alpha}(|k| = N)\) is determined by

\[
(a_{i_0}^{(\alpha} - a_{i_0}^{(\beta} - k_{i_0}) u_{N,k}^{(\alpha} + p_{i_0}^{\beta} = 0.
\]

**Proof.** It is easy to see

\[
Q_i(x) = \left( I + \sum_{|k| = N} U_N^{(a, \beta)} x^k \right)^{-1} P_i(x) \left( I + \sum_{|k| = N} U_N^{(a, \beta)} x^k \right)
\]

\[
- \left( I + \sum_{|k| = N} U_N^{(a, \beta)} x^k \right)^{-1} \left( \sum_{|k| = N} k_i U_N^{(a, \beta)} x^k \right).
\]
Comparing the coefficients of \(x^k\), \(|k|<N\) or \(|k|=N\), we obtain (13) and
\[
Q_{t,k} = P_{t,k} + [A_t, U_{N,k}^{(a,\beta)}] - k_t U_{N,k}^{(a,\beta)},
\]
with \(|k|=N\), \(i=1, \ldots, n\). Since \(A_t\) is lower triangular, we can easily derive (14) and
\[
q^\alpha_{tk} = p^\alpha_{tk} + (a^\alpha_t - a^\beta_t - k_t)u^\beta_{nk}, \quad |k|=N, \quad i=1, \ldots, n,
\]
from (15).

Now suppose \(k(|k|=N)\) satisfies
\[
a^\alpha_{i_0} - a^\beta_{i_0} - k_{i_0} \neq 0
\]
for some \(i_0\).

In order to prove that \(u^\beta_{nk}\) can be chosen so that \(q_{tk}=0\) for all \(i=1, \ldots, n\), it is sufficient to show the relation
\[
(a^\alpha_{i_0} - a^\beta_{i_0} - k_{i_0})p^\beta_{i_0k} - (a^\alpha_t - a^\beta_t - k_t)p^\beta_{tk} = 0, \quad i=1, \ldots, n
\]
holds, which will be called the compatibility condition. It will be shown that the condition (17) can be derived from the integrability condition, which will complete the proof of Proposition 1.

Let us calculate the \((\alpha, \beta)\) component of the integrability condition
\[
k_t P_{t_0,k} - k_{t_0} P_{t,k} + \sum_{k' + k'' = k} [P_{t_0,k'}, P_{t,k''}] = 0.
\]
For the third term of (11)\(_{t_0, t, k}\), we have
\[
\sum_{k' + k'' = k} [P_{t_0,k'}, P_{t,k''}] = [A_{t_0}, P_{t,k}] - [A_t, P_{t_0,k}] + \sum_{k', k''} [P_{t_0,k'}, P_{t,k''}].
\]

The \((\alpha, \beta)\) component of the first term of (18) is expressed as
\[
[A_{t_0}, P_{t,k}]^\alpha\beta = (a^\alpha_{i_0} - a^\beta_{i_0})p^\beta_{ik} + \sum_{\xi<\alpha} a^\alpha_{i_0} p^\beta_{ik} - \sum_{\xi' \geq \beta} \alpha_{i_0} p^\beta_{ik},
\]
because \(A_{t_0}\) is lower triangular. In this relation, if \(a^\alpha_{i_0} p^\beta_{ik} \neq 0\) for some \(\xi<\alpha\), then we have \(a^\alpha_j - a^\beta_j = 0\) for all \(j\) and \(a^\alpha_j - a^\beta_j - k_j = 0\) for all \(j\), by the assumption of the proposition that the system considered is 'reduced' with respect to \((k, (\tau, \delta))\) both for the cases when \(k=0, (\tau, \delta)\) arbitrary and when \(|k|=N\), \((\tau, \delta)<(\alpha, \beta)\). Hence \(a^\alpha_j - a^\beta_j - k_j = 0\) for all \(j\), which is a contradiction to (16). Therefore, \(a^\alpha_{i_0} p^\beta_{ik} = 0\) for all \(\xi<\alpha\). For the same reason, \(a^\alpha_{i_0} p^\beta_{ik} = 0\) for all \(\xi>\beta\). Thus we have
\[
[A_{t_0}, P_{t,k}]^\alpha\beta = (a^\alpha_{i_0} - a^\beta_{i_0})p^\beta_{ik},
\]
and analogously
\[
[A_t, P_{t_0,k}]^\alpha\beta = (a^\alpha_{i_0} - a^\beta_{i_0})p^\beta_{tk}.
\]
Finally, it will be shown that the \((\alpha, \beta)\) component of the third term of (18), i.e.

\[
[P_{t_0,k'}, P_{t',k''}]^\alpha = \sum_{i=1}^{m} p_{i_{0}k'}^\alpha p_{i_{0}k''}^\alpha = \sum_{i=1}^{m} p_{i_{0}k'}^\beta p_{i_{0}k''}^\beta, \quad k' + k'' = k, \quad k', k'' > 0
\]

vanishes. If \(p_{i_{0}k'}^\alpha p_{i_{0}k''}^\beta \neq 0\) for some \(\xi\), then we have

\[a_j^\alpha - a_j^\beta - k' = 0\]

by the assumption of the proposition that our system is 'reduced' with respect to \((k, (\gamma, \delta))\) for \(|k| < N\), \((\gamma, \delta)\) arbitrary. Hence \(a_j^\alpha - a_j^\beta - k = 0\) for all \(j\), which is a contradiction to (16). Therefore \(p_{i_{0}k'}^\alpha p_{i_{0}k''}^\beta = 0\) for all \(\xi\). In the same way, we have \(p_{i_{0}k'}^\beta p_{i_{0}k''}^\beta = 0\) for all \(\xi\). Thus

(21)

\[
[P_{t_0,k'}, P_{t',k''}]^\alpha = 0
\]

for all \(k', k''\) with \(k' + k'' = k, \quad k', k'' > 0\). Therefore, from \((11)_{t_0,t'; k}, (18), (19), (20)\) and (21), we obtain the compatibility condition (17). Thus we have completed the proof of Proposition 1.

By expanding the infinite product of \(\{U_{i_{0}}^{(\alpha, \beta)}(x)\}\) thus determined, we have

**Theorem 2.** One can find a formal power series \(\sum_{k \geq 0} U_k x^k\) with \(\det U_0 \neq 0\), such that the formal substitution \(X = \left(\sum_{k \geq 0} U_k x^k\right) Y\) changes system (1) to a system which has a 'reduced' form.

**Remark 1.** In the induction process above, for sufficiently large \(N\), \(U_{i_{0}}^{(\alpha, \beta)}(x)\) is uniquely determined. Therefore, although \(\sum_{k \geq 0} U_k x^k\) in Theorem 2 is not uniquely determined, it contains only a finite number of undetermined parameters.

**Remark 2.** In Theorem 2, if no two eigenvalues of \(P_{t,0}\) differ by an integer for every \(i\), we can choose \(\sum_{k \geq 0} U_k x^k\) with \(U_0 = I\) such that, by \(X = \left(\sum_{k \geq 0} U_k x^k\right) Y\), (1) is changed to the equation

\[dY = \left(\sum_{i=1}^{n} \frac{P_{t,0}}{x_i} dx_i\right) Y.\]

In this case, \(U_h(k > 0)\) is uniquely determined.

**3.2. Analytic reduction.** As is easily seen, the formal power series \(U(x) = \sum_{k \geq 0} U_k x^k\) in Theorem 2 satisfies the equation
\[ dU(x) = \left( \sum_{i=1}^{n} \frac{P_i(x)}{x_i} dx_i \right) U(x) - U(x) \left( \sum_{i=1}^{n} \frac{Q_i(x)}{x_i} dx_i \right), \]
in the formal sense. If we regard \( U(x) \) as an \( m^2 \)-vector \( u(x) \), (22) is equivalent to
\[ du(x) = \left( \sum_{i=1}^{n} \frac{1}{x_i} (P_i(x) \otimes I - I \otimes^i Q_i(x)) dx_i \right) u(x). \]
Since \( Q_i(x) \) is a polynomial of \( x \), we can apply Theorem 1 to equation (23) to prove the convergence of \( \sum_{k \geq 0} U_k x^k \). Thus, by the definition of a 'reduced' form, we have

**Theorem 3.** Given a completely integrable system (1), we have a convergent series \( U(x) = \sum_{k \geq 0} U_k x^k \) with \( \det U_0 \neq 0 \), such that the transformation \( X = U(x) Y \) takes (1) to
\[ dY = \left( \sum_{i=1}^{n} \frac{Q_i(x)}{x_i} dx_i \right) Y, \]
where
(i) \( Q_i(x) = A_i + \sum_{k>0} Q_{i,k} x^k \) (finite sum), \( A_i = (a_{i}^{**}) \) being lower triangular,
(ii) the \((\alpha, \beta)\) component \( q_i^{\alpha\beta}(x) \) of \( Q_i(x) \) is a monomial of \( x \),
\[ q_i^{\alpha\beta}(x) = q_i^{\alpha\beta} x_1^{h_1} \cdots x_n^{h_n}, \]
with \( k_\mu = a_{\mu}^{\alpha\alpha} - a_{\mu}^{\alpha\beta} \), \( \mu = 1, \ldots, n \). \( q_i^{\alpha\beta}(x) \neq 0 \) implies that \( a_{\mu}^{\alpha\alpha} - a_{\mu}^{\alpha\beta} \) is a nonnegative integer \( k_\mu \) for all \( \mu = 1, \ldots, n \).

§ 4. **Singular transformation.**

Consider the 'reduced' equation (24) in Theorem 3. Let \( L_i \) be a diagonal matrix \( \text{diag} (l_{i1}, \ldots, l_{in}) \). A singular transformation
\[ Y = x_1^{l_{11}} \cdots x_n^{l_{nn}} Z \]
changes (24) to
\[ dZ = \left( \sum_{i=1}^{n} \frac{B_i(x)}{x_i} dx_i \right) Z \]
where
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\[ B_t(x) = x_1^{-L_1} \cdots x_n^{-L_n} Q_t(x) x_1^{L_1} \cdots x_n^{L_n} - L_t, \]

or equivalently

\[ b_t^{\alpha \beta}(x) = q_t^{\alpha \beta}(x) x_1^{L_1 - l_1^\alpha} \cdots x_n^{L_n - l_n^\alpha} - \delta_{\rho \beta} l_t^\rho. \]

Here, \( B_t(x) = (b_t^{\alpha \beta}(x)), Q_t(x) = (q_t^{\alpha \beta}(x)), \) and \( \delta_{\rho \beta} \) denotes the Kronecker symbol.

We shall show that \( b_t^{\alpha \beta}(x) \) becomes constant by choosing nonnegative integers \( \{l_t^\rho\} \) suitably. We classify \( \{a_t^\alpha\}_{\alpha=1, \ldots, m} \) so that \( a_t^\alpha \) and \( a_t^{\beta \rho} \) belong to the same class iff \( a_t^\alpha - a_t^{\beta \rho} \) is an integer. We denote by \([a_t^\alpha]\) the class of \( a_t^\alpha \). For every \( a_t^\alpha \), we denote by \( a_t^{\alpha \beta \rho} \) a member of \([a_t^\alpha]\) which has the minimum real part. Then by taking \( l_t^\rho = a_t^\alpha - a_t^{\beta \rho} \), \( b_t^{\alpha \beta}(x) \) becomes a constant \( b_t^{\alpha \beta} \), by virtue of the properties (i) and (ii) in Theorem 3. Thus we have proved

**Theorem 4.** By a change of variables

\[ Y = x_1^{L_1} \cdots x_n^{L_n} Z \]

with \( L_t = \text{diag}(l_1, \ldots, l_n) \), \( l_t^\rho \) being nonnegative integer, the 'reduced' equation (24) in Theorem 3 is transformed to

\[ dZ = \left( \sum_{i=1}^{n} \frac{B_t}{x_i} dx_i \right) Z \]

where \( B_t \) is a constant matrix given by

\[ B_t = A_t - L_t + \sum_{k>0} Q_{t,k} \quad \text{(finite sum)} \]

and \( [B_t, B_j] = 0 \), \( i, j = 1, \ldots, n \).

**Remark 3.** \( L_t \) in Theorem 4 is not uniquely determined but is unique up to integers in the following sense: Let \( L_t \) and \( L_t' \) be two diagonal matrices satisfying the conditions of Theorem 4, then \( l_t^\rho - l_t'^\rho = l_t^\sigma - l_t'^\sigma \) for all \( \alpha, \beta \) with \([a_t^\alpha] = [a_t^{\beta \rho}]\).

**§ 5. Elimination of nonessential variables.**

In this section, we consider the special case when \( P_i(x)/x_i \) is holomorphic at \( x=0 \) for \( \nu < i \leq n \). In order to eliminate the variables \( x_{i+1}, \ldots, x_n \), we consider a sequence of transformations \( \{X = U_n(x) Y\}_{n=1,2,\ldots} \) where

\[ U_n(x) = I + \sum_{|k|=N, k>0} U_{N,k} x^k, \]

with \( \hat{k} = (k_{i+1}, \ldots, k_n) \). As an induction process we have

**Proposition 2** (Induction process). Assume that the completely integrable system
\[ dX = \left( \sum_{i=1}^{n} \frac{P_i(x)}{x_i} \right) dx, \quad P_i(x) = \sum_{k \geq 0} P_{i,k} x^k, \]

satisfies the conditions:

(i) \( P_i(x)/x_i \) is holomorphic at \( x=0 \) for \( \nu < i \leq n \),

(ii) \( P_{i,k} = 0 \) for \( i \leq \nu, 0 < |k| < N, \hat{k} > 0 \) and for \( i > \nu, 0 \leq |k| < N \).

Then we can choose a suitable substitution \( X = U_N(x)Y \) with

\[ U_N(x) = I + \sum_{|k|=N, \hat{k}>0} U_{N,k} x^k, \]

such that the consequent system

\[ dY = \left( \sum_{i=1}^{n} \frac{Q_i(x)}{x_i} \right) dx, \quad Q_i(x) = \sum_{k \geq 0} Q_{i,k} x^k \]

satisfies the conditions:

(i)' \( Q_i(x)/x_i \) is holomorphic at \( x=0 \) for \( \nu < i \leq n \),

(ii)' \( Q_{i,k} = 0 \) for \( i \leq \nu, 0 < |k| < N, \hat{k} > 0 \) and for \( i > \nu, 0 \leq |k| < N \),

(iii)' \( Q_i(x_1, \cdots, x_i, 0, \cdots, 0) = P_i(x_1, \cdots, x_i, 0, \cdots, 0) \) for \( i \leq \nu \).

Proof. The conditions (i)', (iii)' and the condition (ii)' for \( i \geq \nu, 0 < |k| < N, \hat{k} > 0 \) and for \( i > \nu, 0 \leq k < N \) are immediately verified by

\[ \sum_{i=1}^{n} \frac{Q_i(x)}{x_i} dx = U_N(x)^{-1} \left( \sum_{i=1}^{n} \frac{P_i(x)}{x_i} dx \right) U_N(x) - U_N(x)^{-1} dU_N(x). \]

Therefore, we have only to show that by choosing the values of \( \{U_{N,k}\} \), (ii)' holds for \( i \leq \nu, |k| = N, \hat{k} > 0 \) and for \( i > \nu, 0 \leq k < N \). By (25), we have

\[ Q_{i,k} = P_{i,k} + [P_{i,0}, U_{N,k}] - k_i U_{N,k}, \quad i=1, \cdots, n, |k| = N, \hat{k} > 0. \]

For all \( k \) \( (|k| = N, \hat{k} > 0) \) let \( i(k) \) be an index satisfying \( k_{i(k)} \neq 0 \), and determine \( U_{N,k} \) by

\[ U_{N,k} = -\frac{1}{k_{i(k)}} P_{i(k),k}. \]

For \( U_{N,k} \) thus determined, we can prove \( Q_{i,k} = 0 \) for \( i = 1, \cdots, n, |k| = N, \hat{k} > 0 \), by the integrability condition (11).

Q.E.D.

By this proposition, one determines a sequence of transformations \( \{X = U_N(x)Y\}_{N=1,2,\ldots} \) which formally reduces (1) to

\[ dY = \left( \sum_{i=1}^{n} \frac{P_i(x_1, \cdots, x_i, 0, \cdots, 0)}{x_i} dx \right) Y. \]
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The convergence of the formal power series

\[ U(x) = U_1(x)U_2(x) \cdots U_N(x) = I + \sum_{k>0,\hat{k}>0} U_kx^k \]

can be proved by Theorem 1 in an analogous way as in Section 3. Thus we have

**Theorem 5.** Let a completely integrable system (1) be given. If \( P_i(x)/x_i \) is holomorphic at \( x=0 \) for \( \nu<i \leq n \), then (1) can be changed to the system

\[ dY = \left( \sum_{i=1}^{\nu} \frac{P_i(x_1, \cdots, x_\nu, 0, \cdots, 0)}{x_i} \right) dx_i Y \]

by a substitution \( X = U(x)Y \) where \( U(x) \) is invertible and holomorphic at \( x=0 \) of the form

\[ U(x) = I + \sum_{k>0,\hat{k}>0} U_kx^k. \]

**Remark 4.** It must be noted that, in the above substitution \( U(x) \), we have \( U_k = 0 \) for \( k_{\nu+1} = \cdots = k_n = 0 \). This fact plays an important role in applications.

§ 6. Main theorems.

Combining Theorems 3 and 4, we have

**Theorem 6.** Let a completely integrable system (1) be given, where \( P_i(x) \), \( i = 1, \cdots, n \), is holomorphic at \( x=0 \). Then we have a matrix \( U(x) \) invertible and holomorphic at \( x=0 \) and a diagonal matrix \( L_i, i=1, \cdots, n \), whose components are nonnegative integers, such that the transformation

\[ X = U(x)x_1^{L_1} \cdots x_n^{L_n}Z \]

changes (1) to

\[ dZ = \left( \sum_{i=1}^{n} \frac{B_i}{x_i} dx_i \right) Z \]

where \( B_i, i=1, \cdots, n \), is a constant matrix satisfying \( [B_i, B_j] = 0 \) for all \( i, j = 1, \cdots, n \).

The calculation of the matrices \( B_i, L_i \) and the coefficients in the power series for \( U(x) \) can be concretely performed by algebraic operations.

Combining Theorems 3, 4 and 5, we have

**Theorem 7.** Let a completely integrable system (1) be given, where \( P_i(x) \), \( i = 1, \cdots, n \), is holomorphic at \( x=0 \). If further \( P_i(x)/x_i, i=\nu+1, \cdots, n \), is holomorphic at \( x=0 \), then we have two matrices \( V(x), W(x_1, \cdots, x_\nu) \) invertible and
holomorphic at \( x=0 \) having the form

\[
V(x) = I + \sum_{k>0, k_{p+1}+\cdots+k_{n}>0} V_k x^k
\]

\[
W(x_1, \ldots, x_\nu) = W_0 + \sum_{k_1+\cdots+k_\nu>0} W_{k_1,\ldots,k_\nu} x_1^{k_1} \cdots x_\nu^{k_\nu},
\]

and a diagonal matrix \( L_i, i=1, \ldots, \nu \), whose components are nonnegative integers, such that the transformation

\[
X = U(x)W(x_1, \ldots, x_\nu) Z
\]

can be concretely carried out by algebraic operations.

**Remark 5.** As an immediate consequence of Theorem 7, we have the following theorem which is equivalent to a result obtained by R. Gérard in an abstract way ([13]).

**Theorem.** A completely integrable system

\[
dX = \left( \sum_{i=1}^{\nu} \frac{P_i(x)}{x_i}dx_i + \sum_{i=\nu+1}^{n} \hat{P}_i(x)dx_i \right) X,
\]

with \( P_i(x), i=1, \ldots, \nu \), \( \hat{P}_i(x), i=\nu+1, \ldots, n \), holomorphic at \( x=0 \), has a fundamental matrix solution of the form

\[
U(x)x_1^{L_1} \cdots x_\nu^{L_\nu} Z
\]

Here \( U(x) \) is a matrix invertible and holomorphic at \( x=0 \), \( L_i \) is a diagonal matrix whose components are nonnegative integers and \( B_i \) is a constant matrix with \([B_i, B_j]=0\) for all \( i, j=1, \ldots, \nu \).

**References**


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