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An Extension of Hölder's Theorem Concerning the Gamma Function

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1. Introduction.

O. Hölder proved [3] that Euler's Gamma function $\Gamma(x)$ cannot satisfy any algebraic differential equation whose coefficients are rational functions (i.e. any equation of the form $f(x, y, y', \cdots, y^{(n)})=0$, where n is a nonnegative integer and where f is a polynomial in $y, y', \cdots, y^{(n)}$ whose coefficients are rational functions.) Other proofs of this theorem were given by F. Hausdorff [2], E. H. Moore [7], A. Ostrowski [9], [10], and B. Levin [5]. A natural question is thus raised: for which other fields K of meromorphic functions on the plane will it be true that $\Gamma(x)$ cannot satisfy any algebraic differential equation whose coefficients belong to K? By carefully examining Hausdorff's proof of Hölder's theorem in [2], we isolate four properties of a field (which we call the *Hausdorff Properties* (see Part A below)), such that $\Gamma(x)$ cannot satisfy any algebraic differential equation whose coefficients belong to any field which possesses these properties.

In Part B, we produce fields which possess the Hausdorff Properties. The first example is the field of all meromorphic functions $y(x)$ on the plane, whose Nevanlinna characteristic (see [8; p. 12]) satisfies $T(r, y)=o(r)$ as $r\to+\infty$. This field contains all entire and meromorphic functions whose order of growth is less than one, as well as certain functions whose order of growth is equal to one (i.e. those of minimal type). We remark here that this field contains functions which cannot satisfy any algebraic differential equation whose coefficients are rational functions (see [4; p. 332]). We remark further that since $T(r, \Gamma'/\Gamma)=0(r)$ as $r\to+\infty$ (see Part C), $\Gamma(x)$ obviously satisfies an algebraic differential equation whose coefficients belong to the field of meromorphic functions y for which $T(r, y)=0(r)$ as $r\to+\infty$. We then construct more extensive fields possessing the Hausdorff Properties, as follows: If G is a set of meromorphic functions on the plane for which there exists a fixed sequence $\{r_n\}$ of real numbers tending to $+\infty$ such that for each function $y(x)$ in G, $T(r_n, y)/r_n\to 0$ as $n\to\infty$, then we show that G is contained in a field with the Hausdorff Properties. (Such fields can contain meromorphic functions of infinite order of growth, since G can contain such functions.)

Using transcendence degree arguments, we then show (§ 10) that if K is a field with the Hausdorff Properties, then $\Gamma(x)$ cannot satisfy any algebraic differential equation each of whose coefficients is a meromorphic solution of some algebraic differential equation with coefficients in K.

In Part C, we consider the class of difference equations,

$$y(x+1)-y(x)=R(x), \tag{1}$$

where $R(x)$ is a rational function. (When $R(x)=1/x$, the function $y=\Gamma'/\Gamma$ satisfies (1).) In the course of proving our results in Part B, we have shown (see § 8), that the only meromorphic solutions of an equation of the form (1) with the property that $T(r,y)=o(r)$ as $r\to+\infty$, are rational functions. Thus for many rational functions $R(x)$, the equation (1) has no meromorphic solutions with the property that $T(r,y)=o(r)$ as $r\to+\infty$ (e.g. if $R(x)$ is a rational function which vanishes at ∞ but is not identically zero, and for which no two distinct poles of R differ by an integer (see [2; p. 244]).) However, we show in § 16 that for any rational function $R(x)$, the equation (1) always has a meromorphic solution y with $T(r,y)=0(r)$ as $r\to+\infty$.

Part A The Hausdorff Properties.

If one carefully examines the proof of Hausdorff's Theorem I on pp. 244–246 of [2], it is easy to see that the proof is valid for algebraic differential equations $f=0$, having coefficients in a field K of meromorphic functions on the plane, if K possesses the following properties: (i) K contains the field of rational functions, (ii) if $A(x)$ belongs to K, then the function $A_1(x)=A(x+1)$ also belongs to K, and (iii) if $B(x)$ belongs to K and if $B(x+1)-B(x)$ is a rational function, then $B(x)$ must be a rational function. (Conditions (i) and (ii) guarantee that the algebraic differential equation $g=0$, constructed on p. 245 of [2], has coefficients which belong to K. Condition (iii) guarantees that if a coefficient $B(x)$ in f satisfies condition (2) on page 245 of [2] (i.e. that $B(x)-B(x+1)$ is a linear combination, with constant coefficients, of φ and its derivatives, where φ is a rational function vanishing at ∞ and not having two distinct poles which differ by an integer), then $B(x)$ must be a constant. This will be true since by (iii), $B(x)$ must be a rational function and hence by Hausdorff's lemma [2; p. 244], $B(x)$ must be a constant.) If one carefully examines the proof of Hausdorff's Theorem III on pp. 246–247 of [2], it is easy to see that the proof is valid for algebraic differential equations $f=0$, having coefficients in a field K of meromorphic functions, if K possesses the property that the derivative of each element of K also belongs to K. (This property guarantees that the algebraic differential equation $g=0$, constructed on p. 246 of [2] by differentiating f implicitly with respect to x, also has coefficients belonging to K.)

In view of these facts, we make the following definition.

2. Definition. If K is a field of meromorphic functions on the plane, we will say that K has the *Hausdorff Properties*, if K is a differential field (i.e. the derivative of each element of K also belongs to K), and if K satisfies Conditions (i), (ii) and (iii) listed above.

Thus from the previous discussion, we have the following results.

3. Theorem. *Let K be a field with the Hausdorff Properties. Let $\varphi(x)$ be a rational function which vanishes at ∞ and which does not have two distinct poles differing by an integer. Then a meromorphic function $y(x)$ on the plane, which satisfies the difference equation $y(x+1)-y(x)=\varphi(x)$, cannot satisfy any algebraic differential equation whose coefficients belong to K, unless $\varphi(x)$ is identically zero.*

4. Theorem. *Let K be a differential field of meromorphic functions on the plane. If a meromorphic function $y(x)$ satisfies an algebraic differential equation whose coefficients belong to K, then the function also satisfies such an equation which is homogeneous (i.e. an equation $f=0$, whose coefficients belong to K, such that all terms in f having nonzero coefficients are of the same total degree in $y, y', \cdots, y^{(n)}$).*

Now the function $y(x)=\Gamma'(x)/\Gamma(x)$ satisfies the difference equation $y(x+1)-y(x)=1/x$. Clearly, if $\Gamma(x)$ satisfied a homogeneous algebraic differential equation with coefficients in K, then $\Gamma'(x)/\Gamma(x)$ would satisfy an algebraic differential equation with coefficients in K. Thus, from the above two theorems, we obtain the following result.

5. Theorem. *Let K be a field with the Hausdorff Properties. Then $\Gamma(x)$ cannot satisfy any algebraic differential equation whose coefficients belong to K.*

Part B Fields with the Hausdorff Properties.

6. Notation. For a meromorphic function $y(x)$ on the plane, we will use the standard notation for the Nevanlinna functions $m(r, y)$, $N(r, y)$ and $T(r, y)$ introduced in [8; pp. 6, 12]. We will also use the notation $n(r, y)$ to denote the number of poles (counting multiplicity) of $y(x)$ in $|x| \leq r$.

7. Lemma. (a) *Let $y(x)$ be meromorphic on the plane. Let α be a complex number, and let $u(x)=y(x+\alpha)$. Then for any $a>1$, $T(r, u)=0(T(ar, y))$ as $r \to +\infty$.*

(b) *Let $y(x)$ be meromorphic on the plane. Then for any $a>1$, $T(r, y')=0(T(ar, y))$ as $r \to +\infty$.*

(c) *Let $y(x)$ be a meromorphic function on the plane, such that $y(x+1)-y(x)$ is a rational function. Suppose, in addition, that there exists a sequence $\{r_n\}$ of real numbers tending to $+\infty$, such that $\{T(r_n, y)/r_n\}$ tends to 0 as $n \to \infty$. Then*

$y(x)$ must be a rational function.

Proof. Part (a). Let $b>1$ be such that $b^3=a$. By [6; pp. 372–373], there exist entire functions y_1 and y_2 such that $y=y_1/y_2$, and for $j=1, 2$,

$$T(r, y_j)=0(T(br, y)) \qquad \text{as } r\to+\infty. \tag{2}$$

By [8; p. 24], for $r>0$ and $j=1, 2$,

$$\log^+ M(r, y_j)\leq((b+1)/(b-1))T(br, y_j), \tag{3}$$

where $M(r, y_j)$ is the maximum modulus of y_j on $|x|=r$. Now if we set $u_j(x)=y_j(x+\alpha)$, then clearly,

$$\log^+ M(r, u_j)\leq\log^+ M(br, y_j) \qquad \text{for } r\geq|\alpha|/(b-1), \tag{4}$$

and $u=u_1/u_2$. Since $T(r, u_j)\leq\log^+ M(r, u_j)$ for all $r>0$ by [8; p. 24], it follows easily from (2), (3) and (4), that $T(r, u_j)=0(T(ar, y))$ as $r\to+\infty$ for $j=1, 2$. Since $u=u_1/u_2$, Part (a) now follows immediately using [8; p. 25].

Part (b). From [8; p. 104], we have $T(r, y')\leq 3T(r, y)$ for all $r>0$ with the exception of a set of finite measure. If σ is the measure of the exceptional set, then for any $r>\sigma/(a-1)$, the interval $[r, ar]$ cannot lie completely in the exceptional set, and since $T(t, y)$ is an increasing function of t, it thus follows that $T(r, y')\leq 3T(ar, y)$ for all sufficiently large r, proving Part (b).

Part (c). Before beginning the proof, we make some elementary observations. First, if λ is a complex number, and if for $r>0$, we let $d(r, \lambda)$ be the number of elements of the set $\{\lambda+n: n=1, 2, \cdots\}$ which lie in the disk $|x|\leq r$, then

$$d(r, \lambda)/r\to 1 \qquad \text{as } r\to+\infty. \tag{5}$$

(Let $\lambda=a+ib$, and observe that for r sufficiently large, $d(r, \lambda)$ is the greatest integer less than or equal to $(r^2-b^2)^{1/2}-a$.) Similarly, if $d_1(r, \lambda)$ is the number of elements in the set $\{\lambda-n; n=1, 2, \cdots\}$ which lie in $|x|\leq r$, then,

$$d_1(r, \lambda)/r\to 1 \qquad \text{as } r\to+\infty. \tag{6}$$

We also require the following inequality for an arbitrary meromorphic function $v(x)$:

$$n(r, v)\leq n(0, v)+N(2r, v)/\log 2 \qquad \text{for } r\geq 1. \tag{7}$$

(This follows easily from the definition of $N(r, v)$.)

Now let $y(x)$ be a meromorphic function such that,

$$y(x+1)-y(x)=R(x), \tag{8}$$

where $R(x)$ is rational, and for which there is a sequence $\{r_n\}$ tending to $+\infty$ such that,

$$(9) \qquad T(r_n, y)/r_n \to 0 \qquad \text{as } n \to \infty.$$

Let $L>0$ be so large that all finite poles of $R(x)$ lie in $|x|<L$. We assert that,

(10) All finite poles of y lie in $|x|<L$.

Assume on the contrary, that y has a pole λ with $|\lambda| \geq L$. If the real part of λ is nonnegative, then from (8) it follows that y has poles at $\lambda+1, \lambda+2, \cdots$. Thus we have

$$(11) \qquad n(r, y) \geq d(r, \lambda) \qquad \text{for all } r \geq 0.$$

But in view of (7) and (9), setting $t_n = r_n/2$, we have $n(t_n, y)/t_n \to 0$ as $n \to \infty$, which contradicts (5) and (11). If the real part of λ is negative, then from (8), y has poles at $\lambda-1, \lambda-2, \cdots$, and we get a similar contradiction using (6). This proves (10).

From (10) it follows that y has only finitely many poles in the finite plane. Thus if $Q(x)$ is the sum of the principal parts of $y(x)$ at these poles, then $E=y-Q$ is entire. Let $B(x)=Q(x+1)-Q(x)$, so that,

$$(12) \qquad E(x+1)-E(x)=R(x)-B(x).$$

Since E is entire, $R(x)-B(x)$ must be a polynomial. It is then easy to see that there is a polynomial $Q_1(x)$ such that

$$(13) \qquad Q_1(x+1)-Q_1(x)=B(x)-R(x).$$

Hence if we set $H=Q-Q_1$, then H is rational and $H(x+1)-H(x)=R(x)$. Thus if we set,

$$(14) \qquad E_1=y-H,$$

then we have from (8), that

$$(15) \qquad E_1(x+1) \equiv E_1(x).$$

We observe that E_1 must be entire, for if E_1 had a pole, then it would have infinitely many poles by (15), which would obviously contradict (10). Let $c=E_1(0)$. We claim that,

$$(16) \qquad E_1(x) \equiv c.$$

If we assume the contrary, then $h(x)=1/(E_1(x)-c)$ is a meromorphic function. In view of (15), h has poles at all the positive integers, so that

(17) $$n(r, h) \geq d(r, 0) \qquad \text{for all } r \geq 0.$$

Since $H+c$ is a rational function, it follows from (14) and [8; pp. 15, 16], that

(18) $$T(r, h) \leq T(r, y) + 0(\log r) \qquad \text{as } r \to +\infty.$$

From (9), we thus have $T(r_n, h)/r_n \to 0$ as $n \to \infty$, and hence from (7), if $t_n = r_n/2$, we have $n(t_n, h)/t_n \to 0$ as $n \to \infty$. This obviously contradicts (17) and (5), proving (16). Since E_1 is a constant function, it follows from (14) that y is a rational function, proving Part (c).

Remark. We point out here that in the process of proving Part (c), we also proved the following result. If $y(x)$ is a meromorphic function having only finitely many poles, and satisfying a difference equation $y(x+1)-y(x)=R(x)$, where R is rational, then y can be written as the sum of a rational function and an entire function of period 1 (i.e. (14) and (15) hold).

8. Proposition. *Let K be the set of all meromorphic functions $y(x)$ on the plane such that $T(r, y)=o(r)$ as $r \to +\infty$. Then K is a field with the Hausdorff Properties.*

Proof. The set K is clearly a field which contains all the rational functions (see [8; pp. 15, 16]). From § 7 (a), (b), (c), it follows easily that K has the Hausdorff Properties.

9. Theorem. *Let G be a set of meromorphic functions on the plane for which there is a fixed sequence $\{r_n\}$ of real numbers tending to $+\infty$ such that for each y in G, $T(r_n, y)/r_n \to 0$ as $n \to \infty$. Then G is contained in a field of meromorphic functions which has the Hausdorff Properties.*

Proof. Before constructing the field, we first prove some preliminary results. Let $u(x)$ be a meromorphic function on the plane, and assume there is a sequence $\{t_n\}$ of real numbers tending to $+\infty$, such that

(19) $$T(t_n, u)/t_n \to 0 \qquad \text{as } n \to \infty.$$

Let $s_n = (2/3)t_n$ for $n=1, 2, \cdots$. We assert that if we set $v_k(x)=u(x+k)$, then for each $k=0, 1, \cdots$,

(20) $$T(s_n, v_k)/s_n \to 0 \qquad \text{as } n \to \infty.$$

Let k be given. Applying § 7(a) with $a=3/2$, we have $T(r, v_k)=0(T(3r/2, u))$ as $r \to +\infty$. It is easily seen that (20) now follows from (19).

We now assert that for each $k=0, 1, \cdots$, we have,

(21) $$T(s_n, u^{(k)})/s_n \to 0 \qquad \text{as } n \to \infty.$$

By § 7(b), it is clear that for any $a>1$, we have $T(r, u^{(k)})=0(T(ar, u))$ as $r\to+\infty$. With the choice $a=3/2$, it is clear that (21) now follows from (19).

Now let K be the field of meromorphic functions obtained by adjoining to the field of rational functions, the set of all functions of the form $y^{(j)}(x+k)$, where y belongs to G, and j and k are nonnegative integers (see [1; pp. 76–77]). If u is an element of K, then by [1; p. 35], u can be written as a rational function (whose coefficients are rational functions of x) in finitely many of the functions $y^{(j)}(x+k)$. From this it easily follows that u' and $u(x+1)$ belong to K. Hence it remains to prove that if u is an element of K such that $u(x+1)-u(x)$ is rational, then u must be rational.

Let y belong to G, and let j and k be nonnegative integers. By hypothesis, there is a sequence $\{r_n\}\to+\infty$ such that $T(r_n, y)/r_n\to 0$ as $n\to\infty$. If $\sigma_n=2r_n/3$, then by (19) and (21), $T(\sigma_n, y^{(j)})/\sigma_n\to 0$ as $n\to\infty$. Thus if $\lambda_n=2\sigma_n/3$ and $h_{jk}(x)=y^{(j)}(x+k)$, then it follows from (19) and (20) that $T(\lambda_n, h_{jk})/\lambda_n\to 0$ as $n\to\infty$ for each j and k. Since u is a rational function (with rational coefficients) in the functions h_{jk}, it easily follows from [8; pp. 15–16], that $T(\lambda_n, u)/\lambda_n\to 0$ as $n\to\infty$. Hence if $u(x+1)-u(x)$ is a rational function, then by § 7(c), u must be rational. Thus K has the Hausdorff Properties.

10. The proof of the following result will be completed in § 14.

Theorem. *Let K be a field of meromorphic functions which has the Hausdorff Properties. Then $\Gamma(x)$ cannot satisfy any algebraic differential equation each of whose coefficients is a meromorphic solution of some algebraic differential equation with coefficients in K.*

11. Notation. Let F be a field of meromorphic functions, and let y be a meromorphic function. If n is a nonnegative integer, we denote by $F(y; n)$, the field obtained by adjoining to F the set of functions $\{y, y', \cdots, y^{(n)}\}$. We denote by $F(y; \infty)$, the field obtained by adjoining to F the set $\{y^{(n)}: n=0, 1, \cdots\}$. If L is an extension field of K, we will denote by $T(L/K)$ the transcendence degree (see [1; p. 99]) of L over K.

12. Lemma. *Let L be a differential field (see § 2) of meromorphic functions. Let y be a meromorphic function such that for some nonnegative integer m, the family $(y, y', \cdots, y^{(m)})$ is algebraically dependent over L (see [1; p. 95]). Then if $\sigma=T(L(y; \infty)/L)$, we have*

$$\sigma\leq m. \tag{22}$$

Proof. We may assume that $y, y', \cdots, y^{(m)}$ are all distinct, for if $y^{(j)}=y^{(k)}$ where $0\leq j<k\leq m$, then clearly $L(y; \infty)=L(y; k-1)$. Thus by [1; Th. 2, p. 98],

there is a subset of $\{y, y', \cdots, y^{(k-1)}\}$ which is a transcendence basis of $L(y; \infty)$ over L, so $\sigma \leq k \leq m$.

We now distinguish two cases.

Case I. y is algebraic over L. Then we assert that,

(23) $y, y', \cdots,$ are all algebraic over L.

If not, let q be the smallest nonnegtive integer such that $y^{(q)}$ is not algebraic over L. Then $q \geq 1$ and $y^{(q-1)}$ is algebraic over L. If $A(u)$ is the minimal polynomial of $y^{(q-1)}$ over L, then differentiating $A(y^{(q-1)}(x))=0$ with respect to x, and using the minimality of $A(u)$, we see that $y^{(q)}$ belongs to the field generated by L and $y^{(q-1)}$. Since $y^{(q-1)}$ is algebraic over L, it follows [1; Prop. 6, p. 85] that $y^{(q)}$ is algebraic over L contradicting the definition of q. Thus (23) holds and hence clearly $\sigma=0$.

Case II. If y is transcendental over L, let q be the maximum nonnegative integer j such that the family $(y, y', \cdots, y^{(j)})$ is algebraically independent over L. Clearly, $q<m$. Since $(y, y', \cdots, y^{(q+1)})$ is algebraically dependent over L, there is a nontrivial polynomial $Q(u_0, u_1, \cdots, u_{q+1})$, with coefficients in L, such that

$$Q(y, y', \cdots, y^{(q+1)})=0. \tag{24}$$

If we choose such a polynomial Q of minimal degree d in u_{q+1}, then by definition of q, we must have $d \geq 1$. Hence differentiating (24) with respect to x, and using the minimality of d, it follows that,

(25) $y^{(q+2)}$ belongs to $L(y; q+1)$,

since $y^{(q+2)}=R_1(y, y', \cdots, y^{(q+1)})$, where R_1 is a rational function with coefficients in L. Differentiating this last relation with respect to x, we see that $y^{(q+3)}=R_2(y, y', \cdots, y^{(q+2)})$, where R_2 is a rational function with coefficients in L. Thus from (25), $y^{(q+3)}$ belongs to $L(y; q+1)$. Continuing this way, we see that for all $j \geq q+2$, $y^{(j)}$ belongs to $L(y; q+1)$. Thus clearly, $L(y; \infty)=L(y; q+1)$, and hence from the definition of q, $\{y, y', \cdots, y^{(q)}\}$ is a transcendence basis of $L(y; \infty)$ over L. Since $q+1 \leq m$, we have $\sigma \leq m$, thus proving the result.

13. In the special case where F is the field of complex numbers, the following result was stated without proof by Moore [7; p. 54 (i)].

Proposition. *Let F be a differential field of meromorphic functions. Let y be a meromorphic function which satisfies an algebraic differential equation each of whose coefficients is a meromorphic solution of some algebraic differential equation with coefficients in F. Then y satisfies an algebraic differential equation with coefficients in F.*

Proof. Clearly, we can assume that $y, y', y'', \cdots$ are all distinct. Let y satisfy an equation of order q, with coefficients $a_1, \cdots, a_\sigma$, each satisfying some algebraic differential equation with coefficients in F. Thus, for each j, there is a nonnegative integer n_j such that,

(26) $(a_j, a'_j, \cdots, a_j^{(n_j)})$ is algebraically dependent over F.

For $1 \leq j \leq \sigma$, let F_j be the field obtained by adjoining to F, the functions $a_j, a_{j+1}, \cdots, a_\sigma$, and all their derivatives. Let $F_{\sigma+1}=F$. Then each F_j is clearly a differential field. By [1; Th. 4, p. 100], we have,

$$T(F_1/F)=\sum_{j=1}^{\sigma} T(F_j/F_{j+1}). \tag{27}$$

But clearly, $F_j=F_{j+1}(a_j; \infty)$, and hence from (26) and the previous result, $T(F_j/F_{j+1}) \leq n_j$. Thus from (27),

$$T(F_1/F) \leq \sigma n_0, \tag{28}$$

where n_0 is the maximum of the numbers n_j for $1 \leq j \leq \sigma$. Since $(y, y', \cdots, y^{(q)})$ is algebraically dependent over F_1, we have by the previous result, that

$$T(F_1(y; \infty)/F_1) \leq q. \tag{29}$$

Thus from (28) and (29), $T(F_1(y; \infty)/F) \leq p$, where $p=\sigma n_0+q$. Thus the family $(y, y', \cdots, y^{(p)})$ cannot be algebraically independent over F by [1; Th. 2, p. 98], and hence y satisfies an algebraic differential equation of order $\leq p$, whose coefficients belong to F.

14. Proof of the Theorem in § 10. This follows immediately from the results in § 13 and 5.

15. *Remark.* By § 5, the Hausdorff Properties for a field K are sufficient to guarantee that $\Gamma(x)$ cannot satisfy any algebraic differential equation whose coefficients belong to K. However, we can now easily show that the Hausdorff Properties are not necessary for this to be true. We need the following result.

Proposition. *Let K be a differential field of meromorphic functions. Let $A(K)$ be the set of meromorphic functions which satisfy some algebraic differential equation with coefficients in K. Then $A(K)$ is a differential field.*

Proof. Let u and v belong to $A(K)$, and let F be the field obtained by adjoining to K, the functions u and v and all their derivatives. Thus if $K_1=K(u; \infty)$, then it follows from § 12, that $T(F/K_1)$ and $T(K_1/K)$ are both finite. Thus $T(F/K)$ is finite, and hence it easily follows that $u+v$, uv, u' and $1/u$ (if $u \neq 0$) belong to $A(K)$, proving the proposition.

By § 10, if K has the Hausdorff Properties, then $\Gamma(x)$ cannot satisfy any algebraic differential equation whose coefficients belong to $A(K)$. However, $A(K)$ fails to have one of the Hausdorff Properties, since it contains nonconstant periodic functions of period one, such as $\exp(2\pi ix)$ and $\sin(2\pi x)$.

Part C The Equation $y(x+1)-y(x)=R(x)$.

15. Lemma. *As* $r\to+\infty$, $T(r,\Gamma'/\Gamma)/r\to 1$.

Proof. The function Γ'/Γ has poles only at the nonpositive integers. Thus we have, $r\leq n(r,\Gamma'/\Gamma)\leq r+1$ for all $r\geq 0$. From this, it easily follows that,

$$r-1\leq N(r,\Gamma'/\Gamma)\leq r-1+\log r \qquad \text{for all } r>0. \tag{30}$$

Since $1/\Gamma$ is an entire function of order 1 (see [11; p. 415]), we have $m(r,\Gamma'/\Gamma)=0(\log r)$ as $r\to+\infty$ by [8; p. 63]. The result now follows immediately.

16. Proposition. *If $R(x)$ is a rational function, then the equation,*

$$y(x+1)-y(x)=R(x), \tag{31}$$

always has a meromorphic solution y such that

$$T(r,y)=0(r) \qquad \textit{as } r\to+\infty.$$

Proof. In view of the partial fraction decomposition of $R(x)$ and the linearity of the operator $y(x+1)-y(x)$, it suffices clearly to prove the result when $R(x)$ is a polynomial, and when $R(x)$ is a function of the form $(x-\alpha)^{-m}$, where m is a positive integer. Clearly, when $R(x)$ is a polynomial, (31) possesses a polynomial solution. Suppose now that $R(x)=(x-\alpha)^{-m}$. Let $u=\Gamma'/\Gamma$. Since $u(x+1)-u(x)=x^{-1}$, clearly, if $v(x)=(-1)^{m-1}u^{(m-1)}(x-\alpha)/(m-1)!$, then we have $v(x+1)-v(x)=(x-\alpha)^{-m}$. In view of the previous result, and § 7(a), (b), it easily follows that $T(r,v)=0(r)$ as $r\to+\infty$, proving the result.

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