Funkcialaj Ekvacioj 18 (1975) 227-244

On the Global Classical Solutions of Nonlinear Wave Equations

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§ 0. Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Points in Ω are denoted by $x=(x_1, x_2, \cdots, x_n)$ and the time variable is denoted by t. Consider the Initial-Boundary Value Problems:

$$(1)\quad \begin{cases} (0,1)\quad u''-\Delta u+u'(\gamma+f(u,u'))=0 & (x,t)\in\Omega\times(0,\infty) \\ (0,2)\quad u(x,0)=u_0(x),\ u'(x,0)=u_1(x) & x\in\Omega \\ (0,3)\quad u=0 \quad \text{on} \quad \partial\Omega & t\in[0,\infty) \end{cases}$$

and

$$(2)\quad \begin{cases} (0,4)\quad \begin{cases} u''-\Delta u+u'(\gamma_0+f_0(u,u',v,v'))=0 \\ v''-\Delta v+v'(\gamma_1+f_1(u,u',v,v'))=0 \end{cases} & (x,t)\in\Omega\times(0,\infty) \\ (0,5)\quad \begin{cases} u(x,0)=u_0(x),\ u'(x,0)=u_1(x) \\ v(x,0)=v_0(x),\ v'(x,0)=v_1(x) \end{cases} & x\in\Omega \\ (0,6)\quad u=v=0 \quad \text{on} \quad \partial\Omega & t\in[0,\infty) \end{cases}$$

where γ, γ_i $(i=0,1)$ are constants and f, f_i $(i=0,1)$ satisfy some assumptions given in § 1.

In this paper the question of the existence of global classical solutions of (1) and (2) is investigated.

Many authors, F. E. Browder [1], J. L. Lions [3], W. A. Strauss [4], W. V. WaHL [9], etc, studied nonlinear evolution equations and wave equations and discussed the global or the local solutions in the generalized or the classical sense. It is comparatively easy to find generalized solutions of equations with monotonous nonlinear terms, but in general even with monotonicity conditions, it seems quite difficult to solve the problems in the classical sense for given initial values.

In the author's previous papar [2], the local existence of a classical solution for the equation

$$(0,7)\quad u''-\Delta u+C_1u^p+C_2(u')^{p_2}+C_3|\nabla u|^{2p_3}=f$$

(C_i $(i=1,2,3)$ are constants, p_i $(i=1,2,3)$ are positive integers and f is some smooth function of $(x,t)\in\Omega\times[0,T)$)

was proved by giving sufficiently smooth initial values. It is scarcely possible to get a global solution for the above equation (0,7) in the case of arbitrary spatial dimension n.

For the special case of (0,7), for example

$$(0,8)\quad u''-\Delta u+u^3=f$$

J. Sather [8] proved the existence of a global classical solution in the case $n\leqq 3$ by giving suitable smooth initial data. His main method of the proof is largely depending on the monotonicity of the term u^3.

However, D. Sattinger [9] introduced the idea of "potential well" and applied to a equation which has no monotonicity condition, for instance,

$$(0,9)\quad u''-\Delta u-u^3=0 \quad (n=3)$$

and proved the global existence of a generalized solution. Roughly speaking, his result is that if the initial data are sufficiently smooth and have suitable small norm, then the Initial-Boundary Value Problem for the equation (0,9) has a global generalized solution.

Quite recently M. Nakao and T. Nanbu and Y. Ebihara [7] succeeded to obtain a global classical solution for the equation of type (0,9) with nonzero energy source function by the method of compactness of successive approximating solutions and the idea of "potential well".

In this paper we shall prove the global existence of classical solutions of the Problems (1) and (2) by giving initial values which are sufficiently smooth and have suitable small norm. The main method of reasoning in the paper is the estimation of successive approximating solutions by choosing a suitable bases (*cf.* [2]) and a simple differential inequality obtained from the original equation.

For example, it is shown that the equation

$$(0,10)\quad u''-\Delta u+\gamma u'+C(u')^3=0 \quad (\gamma>0,\ -\infty<C<\infty)$$

has global classical solution in the arbitrary spatial dimension n. It seems that the idea of "potential well" is not applicable to obtain classical solutions for equations of type (0,10).

This paper has three sections apart from this section. In § 1, we give the auxiliary concepts and state some existence theorems. In § 2, we make the proof of the theorems in § 1 and give some examples. In § 3, we generalize our results to general second order evolution equations and at the end we note some remarks for the first order evolution equations.

§ 1. Lemmas and Theorems

Throughout this paper we assume the function spaces considered are all over real field. The notations are as usual (*e. q.* Lions [3], Mizohata [6]). Now at first we mention some lemmas without proof which will be used later.

Lemma 1. (Sobolev)

Let u, v belong to $H^{\alpha}(\Omega)$, then $(u)^p\cdot(v)^q$ belongs to $H^{\alpha}(\Omega)$ and it holds that

$$\|(u)^p(v)^q\|_{H^{\alpha}(\Omega)}\leqq C(\alpha,p,q,\Omega)\{\|u\|_{H^{\alpha}(\Omega)}\}^p\{\|v\|_{H^{\alpha}(\Omega)}\}^q$$

where α is a positive integer with $\alpha\geqq\left[\frac{n}{2}\right]+1$, and p,q are positive integers.

Lemma 2.

Let p, m be positive integers and Ω be $C^{\left[\frac{n}{2}\right]+1+2mp}$-class bounded domain in $\boldsymbol{R}^n$. Let $L(D)=\sum_{|\alpha|\leqq 2m}A_\alpha D^\alpha$ $(A_\alpha\in\boldsymbol{R})$ with the domain $\mathcal{D}(L(D))=\{\varphi(x)\in\overset{\circ}{H}{}^m(\Omega);\ L(D)\varphi(x)\in L^2(\Omega)\}$ be uniformly elliptic in Ω and coercive on $\overset{\circ}{H}{}^m(\Omega)$ i. e., there exists a positive constant C such that for $\varphi\in\overset{\circ}{H}{}^m(\Omega)$,

$$\langle L(D)\varphi,\varphi\rangle\geqq C\|\varphi\|^2_{\overset{\circ}{H}{}^m(\Omega)}.$$

Then $L^p(D)$ with the domain $\{\varphi(x)\in\overset{\circ}{H}{}^{mp}(\Omega);\ L^p(D)\varphi(x)\in L^2(\Omega)\}$ is also coercive on $\overset{\circ}{H}{}^{mp}(\Omega)$.

We can easily verify this Lemma 2 by the method of induction with respect to p.

Lemma 3.

If a nonnegative function $f(t)$ belonging to $C^1[0,T)$ satisfies

$$\begin{cases} f'(t)\leqq F(t)[-\gamma_0+C\{f(t)\}^p] & \text{for}\quad t\in(0,T)\\ f(0)<\left(\dfrac{\gamma_0}{C}\right)^{1/p} \end{cases}$$

with nonnegative function $F(t)$ which belongs to $C^0[0,T)$ and with positive constants γ_0, C, p, then $f(t)$ should be non-increasing function, therefore, it follows that $f(0)\geqq f(t)$ for any $t\in(0,T)$.

Although this Lemma 3 is obvious, it is widely applicable in the global estimations of the energies of equations of evolution.

The following Lemma 4 plays the most essential roles in this paper. (The proof of this lemma was given in [2].)

Lemma 4.

Let H be a separable Hilbert Space and $\{u_p(t)\}$ be a sequence of mappings $u_p(t)$: $[0,T]\to H$ satisfying

(i) $\sup_p\sup_{t\in[0,T]}\|u_p(t)\|_H<+\infty$

(ii) *For any $t_1,t_2\in[0,T]$, there exist positive constants K,h $(0<h\leqq 1)$ independently of p such that*

$$||u_p(t_1)-u_p(t_2)||_H \leqq K|t_1-t_2|^h$$

then there exist a subsequence $\{u_{p_q}(t)\}$ *of* $\{u_p(t)\}$ *and* $u(t)$ *which belongs to* $\mathcal{E}^0_{[0,T]}[H]$ *such that*

$$u_{p_q}(t) \longrightarrow u(t) \qquad (w) \text{ in } H$$

and the convergence is uniform in t. Where (w) means the weak one as usual.

In this paper we call the numbers γ of (0,1) and γ_i $(i=0,1)$ of (0,4) "*dissipative coefficients*".

In general the classical solutions of (1) and (2) can not exist globally in t and even if they exist globally, they are not always uniformly bounded. Our purpose is to find globally existing and uniformly bounded solutions (We call them "non-growing solutions" in the paper.) of (1) and (2). For this aim, we make two definitions.

Definition 1. A set $S\subset C^0(\bar{\Omega})\times C^0(\bar{\Omega})$ is called a well posed set of the Problem (1) if a pair of function $\{u_0(x), u_1(x)\}$ belongs to S then the Problem (1) with initial values $\{u_0(x), u_1(x)\}$ has a unique non-growing solution.

Definition 2. A set $S\subset C^0(\bar{\Omega})\times C^0(\bar{\Omega})\times C^0(\bar{\Omega})\times C^0(\bar{\Omega})$ is called a well posed set of the Problem (2) if a set of functions $\{u_0(x), u_1(x_1), v_0(x), v_1(x)\}$ belongs to S, then the Problem (2) with initial values $\{u_0(x), u_1(x)\}$ $\{v_0(x), v_1(x)\}$ has one and only one non-growing solution.

Now in the next, we make preparations to construct approximating solutions for the Problem (1), (2).

Let us set $\alpha=\left[\frac{n}{2}\right]+1$, and for the brevity we put $\langle u, (-\Delta)^\alpha v\rangle=(u,v)_\alpha$ for $u, v\in \mathring{H}^\alpha(\Omega)$ and $\langle u, (-\Delta)^{\alpha+1}v\rangle=((u,v))_\alpha$ for $u, v\in\mathring{H}^{\alpha+1}(\Omega)$, and $(u,u)_\alpha=|u|^2_\alpha$, $((v,v))_\alpha=||v||^2_\alpha$.

Then from Lemma 2, $|\cdot|_\alpha, ||\cdot||_\alpha$ define equivalent norms to the norms of the spaces $\mathring{H}^\alpha(\Omega), \mathring{H}^{\alpha+1}(\Omega)$ respectively. From now on we identify $|\cdot|_\alpha, ||\cdot||_\alpha$ with the norms of the spaces $\mathring{H}^\alpha(\Omega), \mathring{H}^{\alpha+1}(\Omega)$ respectively.

Here we put $\{\varphi_j\}$ as eigen functions of the operator $(-\Delta)^{\alpha+3}$ whose domain is $\mathring{H}^{\alpha+3}(\Omega)\cap H^{2(\alpha+3)}(\Omega)$

Since $(-\Delta)^{\alpha+3}$ is coercive on $\mathring{H}^{\alpha+3}(\Omega)$, its Green operator is compact in $L^2(\Omega)$ and therefore $\{\varphi_j\}$ is to be assumed a complete base of $\mathring{H}^{\alpha+3}(\Omega)$ (*cf.* [5]).

Now, we make assumptions for the terms $f(u,u')$ and $f_i(u,u',v,v')$ $(i=0,1)$ in the Probems (1), (2) respectively.

(1,1,1) $f(u,v)$ belongs to $C^{\alpha+2}(\boldsymbol{R}^2)$ and satisfies

$$|(vf(u,v),v)_\alpha|\leqq C|v|^2_\alpha\cdot(|v|^2_\alpha+||u||^2_\alpha)^p$$

for $u\in\mathring{H}^{\alpha+1}(\Omega)$, $v\in\mathring{H}^\alpha(\Omega)$

where the constants C, p are independent of u, v.

(1, 1, 2) For $u(t) \in \mathcal{E}^1_{[0,T]}[\overset{\circ}{H}{}^{\alpha+1}](\Omega) \cap \mathcal{E}^2_{[0,T]}[\overset{\circ}{H}{}^{\alpha}(\Omega)]$, it holds that

$$\left|\left(\frac{\partial}{\partial t}u'f(u,u'),u''\right)_\alpha\right| \leqq C(M,N)\{1+\|u'\|_\alpha^2+|u''|_\alpha^2\}$$

for $t\in[0,T]$. The constant $C(M,N)$ depends on $M>0$, $N>0$ where $M=\max_{t\in[0,T]}\|u(t)\|_\alpha$, $N=\max_{t\in[0,T]}|u'(t)|_\alpha$.

(1, 1, 3) For $u(t) \in \mathcal{E}^2_{[0,T]}[\overset{\circ}{H}{}^{\alpha+1}(\Omega)] \cap \mathcal{E}^3_{[0,T]}[\overset{\circ}{H}{}^{\alpha}(\Omega)]$ it holds that

$$\left|\left(\frac{\partial^2}{\partial t^2}u'f(u,u'),u^{(3)}\right)_\alpha\right| \leqq C(L,M,N)\cdot\{1+\|u''\|_\alpha^2+|u^{(3)}|_\alpha^2\}$$

for $t\in[0,T]$.

Here the constant $C(L,M,N)$ depends on $L>0$, $M>0$, $N>0$, where

$$L=\max_{t\in[0,T]}\|u(t)\|_\alpha,\quad M=\max_{t\in[0,T]}\|u'(t)\|_\alpha,\quad N=\max_{t\in[0,T]}|u''(t)|_\alpha.$$

(1, 2, 1) $f_i(u,v,w,z)$ $(i=0,1)$ belong to $C^{\alpha+2}(\boldsymbol{R}^4)$ and satisfy

$$|(vf_0(u,v,w,z),v)_\alpha| \leqq C_0|v|_\alpha^2(\|u\|_\alpha^2+|v|_\alpha^2+\|w\|_\alpha^2+|z|_\alpha^2)^{p_0}$$
$$|(zf_1(u,v,w,z),z)_\alpha| \leqq C_1|z|_\alpha^2(\|u\|_\alpha^2+|v|_\alpha^2+\|w\|_\alpha^2+|z|_\alpha^2)^{p_1}$$

for u, w $\overset{\circ}{H}{}^{\alpha+1}(\Omega)$, $v, z \in \overset{\circ}{H}{}^{\alpha}(\Omega)$ where the constants C_i, p_i $(i=0,1)$ are independent of u, v, w, z.

(1, 2, 2) For $u(t), v(t) \in \mathcal{E}^1_{[0,T]}[\overset{\circ}{H}{}^{\alpha+1}(\Omega)] \cap \mathcal{E}^2_{[0,T]}[\overset{\circ}{H}{}^{\alpha}(\Omega)]$, it holds that

$$\left|\left(\frac{\partial}{\partial t}\{u'f_0(u,u',v,v')\},u''\right)_\alpha+\left(\frac{\partial}{\partial t}\{v'f_1(u,u',v,v')\},v''\right)_\alpha\right|$$
$$\leqq C(M_0,M_1,N_0,N_1)\{1+\|u'\|_\alpha^2+|u''|_\alpha^2+\|v'\|_\alpha^2+|v''|_\alpha^2\}$$

for $t\in[0,T]$. Here the constant $C(M_0,M_1,N_0,N_1)$ depends on M_i, N_i where $M_0=\max_{t\in[0,T]}\|u(t)\|_\alpha$, $M_1=\max_{t\in[0,T]}\|v(t)\|_\alpha$, $N_0=\max_{t\in[0,T]}|u'(t)|_\alpha$, $N_1=\max_{t\in[0,T]}|v'(t)|_\alpha$.

(1, 2, 3) For $u(t), v(t) \in \mathcal{E}^2_{[0,T]}[\overset{\circ}{H}{}^{\alpha+1}(\Omega)] \cap \mathcal{E}^3_{[0,T]}[\overset{\circ}{H}{}^{\alpha}(\Omega)]$ it holds that

$$\left|\left(\frac{\partial^2}{\partial t^2}\{u'f(u,u',v,v')\},u^{(3)}\right)_\alpha+\left(\frac{\partial^2}{\partial t^2}\{v'f_1(u,u',v,v')\},v^{(3)}\right)_\alpha\right|$$
$$\leqq C(M_0,M_1,N_0,N_1,R_0,R_1)\{1+\|u''\|_\alpha^2+|u^{(3)}|_\alpha^2+\|v''\|_\alpha^2+|v^{(3)}|_\alpha^2\}$$

for $t\in[0,T]$. Here the constant $C(M_0,M_1,N_0,N_1,R_0,R_1)$ depends on M_i, N_i, R_i where $M_0=\max_{t\in[0,T]}\|u(t)\|_\alpha$, $M_1=\max_{t\in[0,T]}\|v(t)\|_\alpha$, $N_0=\max_{t\in[0,T]}\|u'(t)\|_\alpha$, $N_1=\max_{t\in[0,T]}\|v'(t)\|_\alpha$, $R_0=\max_{t\in[0,T]}|u''(t)|_\alpha$, $R_1=\max_{t\in[0,T]}|v''(t)|_\alpha$.

Now we give the initial values in the following spaces.

(1,3) For the Problem (1), $\{u_0(x), u_1(x)\}$ is given in the space $\mathring{H}^{\alpha+3}(\Omega)\times \mathring{H}^{\alpha+2}(\Omega)$.

(1,4) For the Problem (2), $\{u_0(x), u_1(x)\}$, $\{v_0(x), v_1(x)\}$ are given in the space $\mathring{H}^{\alpha+3}(\Omega)\times\mathring{H}^{\alpha+2}(\Omega)$.

Then we can find sequences of numbers satisfying

$$(1,5,1)\quad \sum_{j=1}^{k} a_j\varphi_j = u_k(0) \longrightarrow u_0 \qquad (s) \text{ in } \mathring{H}^{\alpha+3}(\Omega)$$

$$(1,5,2)\quad \sum_{j=1}^{k} b_j\varphi_j = u_k'(0) \longrightarrow u_1 \qquad (s) \text{ in } \mathring{H}^{\alpha+2}(\Omega)$$

$$(1,5,3)\quad \sum_{j=1}^{k} \alpha_j\varphi_j = v_k(0) \longrightarrow v_0 \qquad (s) \text{ in } \mathring{H}^{\alpha+3}(\Omega)$$

$$(1,5,4)\quad \sum_{j=1}^{k} \beta_j\varphi_j = v_k'(0) \longrightarrow v_1 \qquad (s) \text{ in } \mathring{H}^{\alpha+2}(\Omega)$$

where (s) means the strong convergence in these spaces.

Under these preparations, the approximating solutions are constructed as follows. For the Problem (1), we put $u_k(t)=\sum_{j=1}^{k}\lambda_{k,j}(t)\varphi_j$ where the functions $\{\lambda_{k,j}(t)\}$ $(j=1,2,\cdots,k)$ are solutions of the system of ordinary differential equations;

$$(1)'\quad \begin{cases} (u_k'', \varphi_j)_\alpha + ((u_k, \varphi_j))_\alpha + (u_k'\{\gamma + f(u_k, u_k')\}, \varphi_j)_\alpha = 0 & (j=1,2,\cdots,k) \\ \lambda_{k\cdot j}(0)=a_j, \quad \lambda_{k\cdot j}'(0)=b_j & (j=1,2,\cdots,k) \end{cases}$$

As is well known from the theory of ordinary differential equations, $\{\lambda_{k,j}(t)\}$ exist in some interval $[0,\delta_k)$ where $\delta_k>0$ depends on k, and since $f(\cdot,\cdot\cdot)$ is smooth ((1,1,1)), $\{\lambda_{k,j}(t)\}$ belong to $C^4[0,\delta_k)$. (*cf.* [5])

Hence $u_k(t)=\sum_{j=1}^{k}\lambda_{k,j}(t)\varphi_j$ belongs to $\mathcal{E}^4_{[0,\delta_k)}[\mathring{H}^{\alpha+3}(\Omega)]$.

More precisely $u_k(t)$ should belong to $\mathcal{E}^4_{[0,\delta_k)}[\mathring{H}^{\alpha+3}(\Omega)\cap C^\infty(\Omega)]$ from the ellipticity of $(-\Delta)$.

Similarly, for the Problem (2), we put $u_k(t)=\sum_{j=1}^{k}\mu_{k,j}(t)\varphi_j$, $v_k(t)=\sum_{j=1}^{k}\nu_{k,j}(t)\varphi_j$ where the functions $\{\mu_{k,j}(t)\}$, $\{\nu_{k,j}(t)\}$ $(j=1,2,\cdots,k)$ are solutions of the system:

$$(2)\quad \begin{cases} \begin{cases} (u_k'', \varphi_j)_\alpha + ((u_k, \varphi_j))_\alpha + (u_k'\{\gamma_0 + f_0(u_k, u_k', v_k, v_k')\}, \varphi_j)_\alpha = 0 \\ (v_k'', \varphi_j)_\alpha + ((v_k, \varphi_j))_\alpha + (v_k'\{\gamma_1 + f_1(u_k, u_k', v_k, v_k')\}, \varphi_j)_\alpha = 0 \end{cases} \\ \qquad (j=1,2,\cdots,k) \\ \mu_{k,j}(0)=a_j, \quad \mu_{k,j}'(0)=b_j, \quad \nu_{k,j}(0)=\alpha_j, \quad \nu_{k,j}'(0)=\beta_j \\ \qquad (j=1,2,\cdots,k) \end{cases}$$

Then we know $\{u_k(t), v_k(t)\}$ belong to $\mathcal{E}^4_{[0,\varepsilon_k)}[\mathring{H}^{\alpha+3}(\Omega)]$ for some interval $[0,\mathcal{E}_k)$.

Now we have the existence theorems.

Theorem 1.

If the solutions $\{u_k(t)\}$ *of the Problems* (1)$'$ *exist in the interval* $[0, T]$ *independently of* k *and satisfy*

$$\sup_k \sup_{t\in[0,T]} \{|u_k'(t)|_\alpha + \|u_k(t)\|_\alpha\} < +\infty$$

then the Problem (1) *has one and only one classical solution in* $[0, T]$.

Theorem 2.

If the solutions $\{(u_k(t), v_k(t))\}$ *of the Problems* (2)$'$ *exist in the interval* $[0, T]$ *independently of* k *and satisfy*

$$\sup_k \sup_{t\in[0,T]} \{|u_k'(t)|_\alpha + |v_k'(t)|_\alpha + \|u_k(t)\|_\alpha + \|v_k(t)\|_\alpha\} < +\infty$$

then the Problem (2) *has unique classical solution in* $[0, T]$.

Remark The local existence is known by Y. Ebihara [2].

Theorem 3.

If the dissipative coefficient γ *is positive, then the Problem* (1) *has a non-trivial well posed set.*

Theorem 4.

If the dissipative coefficients γ_i $(i=0, 1)$ *are positive, then the Problem* (2) *has a non-trivial well posed set.*

§2. Proofs of the Theorems and Examples

(A) Proof of the Theorem 1.

At first we prove the existence of a classical solution.

From the assumption

$$\sup_k \sup_{t\in[0,T]} \{|u_k'(t)|_\alpha + \|u_k(t)\|_\alpha\} = M < +\infty, \tag{2,1}$$

$u_k(t)$ belongs to $\mathcal{E}^4_{[0,T]}[\mathring{H}^{\alpha+3}(\Omega)]$ for every k.

Now by differentiating the equality of (1)$'$ with respect to t, and summing over j from 1 to k after multiplying $\lambda_{k,j}''(t)$, we have for $t\in(0, T]$

$$\frac{1}{2}\frac{d}{dt}\{|u_k''|_\alpha^2 + \|u_k'\|_\alpha^2\} + \left(\frac{\partial}{\partial t}\{u_k'\{\gamma + f(u_k, u_k')\}\}, u_k''\right)_\alpha = 0.$$

Then it follows from (1, 1, 2) that

$$\frac{d}{dt}\{|u_k''|_\alpha^2 + \|u_k'\|_\alpha^2\} \leqq 2C(M_1, N_1)\{1 + \|u_k'\|_\alpha^2 + |u_k''|_\alpha^2\}$$

where

$$M_1=\sup_k \sup_{t\in[0,T]} \|u_k(t)\|_\alpha, \qquad N_1=\sup_k \sup_{t\in[0,T]} |u_k'(t)|_\alpha.$$

Thus we obtain for $t\in[0,T]$

$$|u_k''(t)|_\alpha^2+\|u_k(t)\|_\alpha^2\leqq C_1(M_1,N_1,T)+C_2(M_1,N_1,T)\{|u_k''(0)|_\alpha^2+\|u_k'(0)\|_\alpha^2\},$$

where C_i $(i=1,2)$ depends on M_1, N_1, T.

Here we show the boundedness of $\{\|u_k'(0)\|_\alpha^2\}$ and $\{|u_k''(0)|_\alpha^2\}$.

From (1, 5, 2), the boundedness of $\{\|u_k'(0)\|_\alpha^2\}$ is obvious.

In the equality of (1)′ we have as $t\to 0$

$$(u_k''(0),\varphi_j)_\alpha+((u_k(0),\varphi_j))_\alpha+(u_k'(0)\{\gamma+f(u_k(0),u_k'(0))\},\varphi_j)_\alpha=0$$

and therefore it follows that

$$|u_k''(0)|_\alpha\leqq|(-\Delta)u_k(0)|_\alpha+|u_k'(0)\{\gamma+f(u_k(0),u_k'(0))\}|_\alpha.$$

Since $u_k(0)\longrightarrow u_0(s)$ in $\mathring{H}^{\alpha+3}(\Omega)$, $u_k'(0)\longrightarrow u_1(s)$ in $\mathring{H}^{\alpha+2}(\Omega)$ ((1, 5, 1), (1, 5, 2)), $\{\|u_k(0)\|_{\mathring{H}^{\alpha+3}(\Omega)}\}$, $\{\|u_k'(0)\|_{\mathring{H}^{\alpha+2}(\Omega)}\}$ are bounded and hence $\{|(-\Delta)u_k(0)|_\alpha\}$ $\{|u_k'(0)\{\gamma+f(u_k(0),u_k'(0))\}|_\alpha\}$ are bounded. In fact, from Sobolev imbedding theorem, the boundedness of $\{|u_k(0)|_{\mathcal{B}^3(\bar\Omega)}\}$ and $\{|u_k'(0)|_{\mathcal{B}^2(\bar\Omega)}\}$ follows immediately and therefore supremums of

$$\left|\frac{\partial^l}{\partial X^{l_1}\partial Y^{l_2}}f(X,Y)\right|_{X=u_k(0),\,Y=u_k'(0)}$$

with $0\leq l=l_1+l_2\leqq\alpha$ are bouned as a matter of course.

Thus for $|a|=\sum a_i\leqq\alpha$, $\{|D^a\{f(u_k(0),u_k'(0))u_k'(0)\}|_{L^2(\Omega)}\}$ are bounded.

From these facts, it yields the boundedness of the right-hand side of the last inequality.

(2, 2) $$\sup_k \sup_{t\in[0,T]} \|u_k'(t)\|_\alpha<+\infty$$

(2, 3) $$\sup_k \sup_{t\in[0,T]} |u_k''(t)|_\alpha<+\infty$$

In the next, after differentiating equality of (1)′ two times we have as above for $t\in(0,T]$,

$$\frac{1}{2}\frac{d}{dt}\{|u_k^{(3)}|_\alpha^2+\|u_k''\|_\alpha^2\}+\left(\frac{\partial^2}{\partial t^2}\{u_k'\{\gamma+f(u_k,u_k')\}\},u_k^{(3)}\right)_\alpha=0.$$

Therefore from (1, 1, 3) and (2, 1)∼(2, 3) it holds that

$$\frac{d}{dt}\{|u_k^{(3)}\|_\alpha^2+\|u_k''\|_\alpha^2\}\leqq 2C(M_1,N_2,L)\{1+\|u_k''\|_\alpha^2+|u_k^{(3)}|_\alpha^2\}$$

and thus we get for $t\in[0,T]$,

$$|u_k^{(3)}(t)|_\alpha^2+\|u_k''(t)\|_\alpha^2 \leqq C_1(M_1, N_2, L, T)+C_2(M_1, N_2, L, T)\cdot\{|u_k^{(3)}(0)|_\alpha^2 + \|u_k''(0)\|_\alpha^2\}$$

where C_i $(i=1,2)$ depends on M_1, N_2, L, T with

$$N_2=\sup_k \sup_{t\in[0,T]} \|u_k'(t)\|_\alpha, \qquad L=\sup_k \sup_{t\in[0,T]} |u_k''(t)|_\alpha.$$

Therefore if we could show the boundedness of $\{|u_k^{(3)}(0)|_\alpha^2\}$ and $\{\|u_k''(0)\|_\alpha^2\}$ we have

$$(2,4) \qquad \sup_k \sup_{t\in[0,T]} \|u_k''(t)\|_\alpha<+\infty$$

$$(2,5) \qquad \sup_k \sup_{t\in[0,T]} |u_k^{(3)}(t)|_\alpha<+\infty.$$

At first we obtain as $t\to 0$ for the differentiated equality of (1)$'$,

$$(u_k^{(3)}(0), \varphi_j)_\alpha+((u_k'(0), \varphi_j))_\alpha+\left(\frac{\partial}{\partial t}\{u_k'(t)\{\gamma+f(u_k(t), u_k'(t))\}\}|_{t=0}, \varphi_j\right)_\alpha=0$$

and from this, it holds that

$$|u_k^{(3)}(0)|_\alpha \leqq |(-\Delta)u_k'(0)|_\alpha+\left|\frac{\partial}{\partial t}\{u_k'(t)\{\gamma+f(u_k(t), u_k'(t))\}\}|_{t=0}\right|_\alpha.$$

From (1,5,2), we obtain the boundedness of $\{|(-\Delta)u_k'(0)|_\alpha\}$.

Using our result (2,3) in the case $t=0$ and (1,5,1), (1,5,2) we have the boundedness of the latter term of this inequality from the same reason as a previous one.

Secondly we show the boundedness of $\{\|u_{kl}''(0)\|_\alpha^2\}$ where kl is the last index which satisfies

$$\langle \varphi_1, \varphi_2, \cdots, \varphi_{kl}\rangle = V(\lambda_1, \lambda_2, \cdots, \lambda_k)$$

here $\langle\cdots\rangle$ is the totality of linear combinations and $V(\cdots)$ means the direct sum of eigen spaces of $\lambda_1, \lambda_2, \cdots, \lambda_k$.

We noted that from ellipticity of Δ, $\{\varphi_j\}$ belong to $\mathring{H}^{\alpha+3}(\Omega)\cap C^\infty(\Omega)$, and so if φ belongs to $\langle \varphi_1, \varphi_2, \cdots, \varphi_{kl}\rangle$, we have

$$\Delta\varphi \in \langle \varphi_1, \varphi_2, \cdots, \varphi_{kl}\rangle.$$

Therefore it follows that

$$-\Delta\{u_{kl}''(0)\} \in \langle \varphi_1, \varphi_2, \cdots, \varphi_{kl}\rangle, \quad \text{and}$$

we may substitute $-\Delta\{u_{kl}''(0)\}$ as φ_j in the equality of (1)$'$ as $t\to 0$. That is, it holds,

$$(u_{kl}''(0), -\Delta\{u_{kl}''(0)\})_\alpha+((u_{kl}(0), -\Delta\{u_{kl}(0)\}))_\alpha$$

$$+(u'_{kl}(0)\{\gamma+f(u_{kl}(0),u'_{kl}(0)\},-\Delta\{u_{kl}(0)\})_\alpha=0$$

and thus, it follows that

$$\|u''_{kl}(0)\|_\alpha\leqq|(-\Delta)^{\frac{\alpha+3}{2}}u_{kl}(0)|_{L^2(\Omega)}+|(-\Delta)^{\frac{\alpha+2}{2}}\{u'_{kl}(0)\{\gamma+f(u_{kl}(0),u'_{kl}(0)\}\}|_{L^2(\Omega)}$$

The right-hand side of this inequality is bounded again by (1, 5, 1), (1, 5, 2). Consequently we have the boundedness of $\{\|u''_{kl}(0)\|_\alpha^2\}$.

From the results (2, 1), (2, 2), $\{u_k(t)\}$ satisfy the assumption of Lemma 4 in the space $\mathring{H}^{\alpha+1}(\Omega)$ with $h=1$ and by combining (2, 2) and (2, 4), $\{u'_k(t)\}$ satisfy the condition in the space $\mathring{H}^{\alpha+1}(\Omega)$ and by (2, 3) and (2, 5), $\{u''_k(t)\}$ satisfy in the space $\mathring{H}^\alpha(\Omega)$.

Therefore by choosing a suitable subsequence $\{u_{k_q}(t)\}$, there exists $u(t)$ which belongs to $\mathcal{E}^1_{[0,T]}[\mathring{H}^{\alpha+1}(\Omega)]\cap\mathcal{E}^2_{[0,T]}[\mathring{H}^\alpha(\Omega)]$ such that

$$(2,6)\qquad \begin{cases} u_{k_q}(t)\Longrightarrow u(t) & (w) \text{ in } \mathring{H}^{\alpha+1}(\Omega)\\ u'_{k_q}(t)\Longrightarrow u'(t) & (w) \text{ in } \mathring{H}^{\alpha+1}(\Omega)\\ u''_{k_q}(t)\Longrightarrow u''(t) & (w) \text{ in } \mathring{H}^{\alpha+1}(\Omega)\end{cases}$$

where $\Longrightarrow$ means uniform convergence in t.

Moreover, since the space $\mathcal{E}^2_{[0,T]}[\mathring{H}^{\alpha+1}(\Omega)]$ is compact in the space $\mathcal{E}^1_{[0,T]}[\mathring{H}^\alpha(\Omega)]$ (by a generalization of Rellich's Theorem), we can assume that

$$u_{k_q}(t)\longrightarrow u(t) \qquad \text{in } \mathcal{E}^1_{[0,T]}[\mathring{H}^\alpha(\Omega)]$$

because of the result (2, 4).

Therefore we may have for $t\in[0,T]$,

$$u'_{k_q}(t)\{\gamma+f(u_{k_q}(t),u'_{k_q}(t))\}\longrightarrow u'(t)\{\gamma+f(u(t),u'(t))\} \quad \text{in } \mathring{H}^\alpha(\Omega).$$

Hence $u(t)$ satisfies for any $\varphi_j\in\{\varphi_j\}$,

$$\begin{cases}(u''(t),\varphi_j)_\alpha+((u(t),\varphi_j))_\alpha+(u'(t)\{\gamma+f(u(t),u'(t))\},\varphi_j)_\alpha=0\\ \qquad\qquad\qquad\qquad\qquad\qquad\text{for } t\in(0,T]\\ u(0)=u_0,\quad u'(0)=u_1.\end{cases}$$

Thus $u(t)$ satisfies

$$u''(t)-\Delta u(t)+u'(t)\{\gamma+f(u(t),u'(t))\}=0$$

for every $t\in(0,T]$ (and for $a.e.\ x$ in Ω).

The ellipticity of Δ yields that $u(t)$ should belongs to $\mathcal{E}^0_{[0,T]}[\mathring{H}^{\alpha+1}(\Omega)\cap H^{\alpha+2}(\Omega)]\cap\mathcal{E}^1_{[0,T]}[\mathring{H}^{\alpha+1}(\Omega)]\cap\mathcal{E}^2_{[0,T]}[\mathring{H}^\alpha(\Omega)]$, because $u''(t)$, and $f(u(t),u'(t))$ are the functions in $\mathcal{E}^0_{[0,T]}[\mathring{H}^\alpha(\Omega)]$.

Consequently, we can see that $u(x,t)$ is a classical solution of the Problem (1).

Secondly we prove the uniqueness.

Let $u(t)$, $U(t)$ be classical solutions with the initial values $\{u_0(x), u_1(x)\}$, $\{U_0(x), U_1(x)\}$ respectively.

Then the functional

$$\varphi(t)=(W'(t),\ W'(t))_{L^2(\Omega)}+(-\Delta W(t),\ W(t))_{L^2(\Omega)}$$

where $W(t)=U(t)-u(t)$, satisfies

$$\begin{aligned}\varphi'(t)&=2\{(W''(t),\ W'(t))_{L^2(\Omega)}+(-\Delta W(t),\ W'(t))_{L^2(\Omega)}\\&=-2(\gamma W(t)+U'(t)f(U(t),U'(t))-u'(t)f(u(t),u'(t)),\ W'(t))_{L^2(\Omega)}\\&=-2\gamma(W(t),\ W'(t))_{L^2(\Omega)}\\&\quad-2(U'(t)f(U(t),U'(t))-u'(t)f(u(t),u'(t)),\ W'(t))_{L^2(\Omega)}.\end{aligned}$$

Since $f(\cdot,\cdot\cdot)$ is smooth, by mean value theorem, we have

$$vf(u,v)-zf(w,z)=F(u,v,w,z)\{(u-w)+(v-z)\}$$

where $F(u,v,w,z)$ is a continuous function of u,v,w,z.

Therefore, it follows that

$$\begin{aligned}\varphi'(t)&=-2\gamma(w(t),w'(t))_{L^2(\Omega)}\\&\quad-2(F(U(t),U'(t),u(t),u'(t))\{W(t)+W'(t)\},\ W'(t))_{L^2(\Omega)}\\&\leqq 2|\gamma|\,|W(t)|_{L^2(\Omega)}\cdot|W'(t)|_{L^2(\Omega)}\\&\quad+2C_0(U,U',u,u')\cdot\{|W(t)|_{L^2(\Omega)}|W'(t)|_{L^2(\Omega)}+|W'(t)|^2_{L^2(\Omega)}\}\end{aligned}$$

where $C_0(U,U',u,u')$ is a constant depending on supremums in $\bar{\Omega}\times[0,T]$ of $|U|, |U'|, |u|, |u'|$.

Consequently, we have for $t\in(0,T]$

$$\begin{aligned}\varphi'(t)&\leqq\text{const}\{|W(t)|^2_{L^2(\Omega)}+|W'(t)|^2_{L^2(\Omega)}\}\\&\leqq\text{const}\{|W'(t)|^2_{L^2(\Omega)}+(-\Delta W(t),\ W(t))_{L^2(Q)}\}\\&=C_1\varphi(t)\end{aligned}$$

Thus we have for $t\in[0,T]$,

$$\varphi(t)\leqq e^{C_1T}\varphi(0).$$

this shows the solution of the Problem (1) is unique.

This completes the proof of the Theorem 1. (q. e. d.)

(B) Proof of the Theorem 2.

The procedure of the proof is completely similar to the one of the Theorem 1.

From the assumptions

$$(2,7)\qquad L=\sup_k\sup_{t\in[0,T]}\{|u_k'(t)|_\alpha+|v_k'(t)|_\alpha+\|u_k(t)\|_\alpha+\|v_k(t)\|_\alpha\}<+\infty$$

and (1,2,2) we have by differentiating equalities of (2)′ and substituting $u_k^{(3)}(t), v_{k_l}^{(3)}(t)$ as φ_j, and summing them,

$$\frac{1}{2}\frac{d}{dt}\{|u_k''|_\alpha^2+|v_k''|_\alpha^2+\|u_k'(t)\|_\alpha^2+\|v_k'(t)\|_\alpha^2\}$$
$$\leqq C(L)\{1+|u_k''|_\alpha^2+|v_k''|_\alpha^2+\|u_k'\|_\alpha^2+\|v_k'\|_\alpha^2\}$$

for $t\in(0, T]$.

From this we have

$$(2,8)\qquad M=\sup_k \sup_{t\in[0,T]} \{|u_k''(t)|_\alpha^2+|v_k''(t)|_\alpha^2+\|u_k'(t)\|_\alpha^2+\|v_k'(t)\|_\alpha^2\}<+\infty$$

by checking the boundedness of $\{|u_k''(0)|_\alpha\}$, $\{|v_k''(0)|_\alpha\}$, $\{\|u_k'(0)\|_\alpha\}$, $\{\|v_k'(0)\|_\alpha\}$.

And further we know

$$(2,9)\qquad N=\sup_{kl} \sup_{t\in[0,T]} \{|u_{kl}^{(3)}(t)|_\alpha^2+|v_{kl}^{(3)}(t)|_\alpha^2+\|u_{kl}''(t)\|_\alpha^2+\|v_{kl}''(t)\|_\alpha^2\}<+\infty$$

by using (1,2,3), and (2,7), (2,8), where kl is the number defined in the proof of the Theorem 1.

Consequently, we get a pair of functions $\{u(t), v(t)\}$ which solves the problem (2).

Secondly we put $\{u(t), v(t)\}$, $\{U(t), V(t)\}$ be solutions with the initial values $\{(u_0, u_1), (v_0, v_1)\}$, $\{(U_0, U_1), (V_0, V_1)\}$ respectively and set

$$\psi(t)=(w(t), w(t))_{L^2(\Omega)}+(W(t), W(t))_{L^2(\Omega)}$$
$$+(-\Delta w(t), w(t))_{L^2(\Omega)}+(-\Delta W(t), W(t))_{L^2(\Omega)}$$

where $w(t)=U(t)-u(t)$, $W(t)=V(t)-v(t)$, then we have the following inequality by the same way of the proof of the Theorem 1.

$$\psi'(t)\leqq \text{const}\cdot\psi(t) \qquad \text{for} \quad t\in(0, T].$$

Uniqueness follows immediately from this. (q. e. d)

(C) Proof of the Theorem 3.

Let us set

$$S=\left\{\{u, v\}\in \mathring{H}^{\alpha+3}(\Omega)\times\mathring{H}^{\alpha+2}(\Omega) \mid \|u\|_\alpha^2+|v|_\alpha^2<\left(\frac{\gamma}{C}\right)^{\frac{1}{p}}\right\}$$

where C, p are the constants in (1,1,1).

Now we verify S is a well posed set of the Problem (1).

If $\{u_0, u_1\}$ belongs to S, $\{u_k(0), u_k'(0)\}$ belongs to S for sufficiently large number $k\geqq k_0$ from the continuity of the functional $\|u_k\|_\alpha^2+|v_k|_\alpha^2$.

And therefore we have for the solutions $\{u_k(t)\}$ $(k\geqq k_0)$ of the Problems (1)′ by a simple calculation,

$$(2,10)\qquad \begin{cases} \dfrac{d}{dt}(|u_k'|_\alpha^2+||u_k||_\alpha^2)\leqq 2|u_k'|_\alpha^2\{-\gamma+C(|u_k'|_\alpha^2+||u_k||_\alpha^2)^p\} \\ \qquad\qquad\qquad\qquad \text{for} \quad t\in(0,\delta_k) \\ (|u_k'(0)|_\alpha^2+||u_k(0)||_\alpha^2)<\left(\dfrac{\gamma}{C}\right)^{\frac{1}{p}}. \end{cases}$$

Consequently by Lemma 2, it follows that $\{u_k(t)\}$ should exist in the interval $[0,\infty)$ and satisfies for arbitrarily fixed positive number T,

$$(2,11)\qquad \sup_{k\geqq k_0}\ \sup_{t\in[0,T]}\{|u_k'(t)|_\alpha^2+||u_k(t)||_\alpha^2\}<\left(\frac{\gamma}{C}\right)^{\frac{1}{p}}$$

Thus by the Theorem 1, we have a function $u(t)$ which solves the Problem (1) in $[0,T]$ and from the uniqueness and the arbitrariness of T, we can extend the existence interval to $[0,\infty)$.

This completes the proof. (q. e. d)

(D) Proof of the Theorem 4.

In the assumption (1, 2, 1), we may assume that $p_0\geqq p_1$, and we put $\gamma=\min\{\gamma_0,\gamma_1\}$, $C=\max\{C_1,C_2\}$.

Now let us set

$$W=\Big\{(\{u_1,u_2\},\{v_1,v_2\})\in(\mathring{H}^{\alpha+3}(\Omega)\times\mathring{H}^{\alpha+2}(\Omega)) \times(\mathring{H}^{\alpha+3}(\Omega)\times\mathring{H}^{\alpha+2}(\Omega))\,| \\ (||u_1||_\alpha^2+|u_2|_\alpha^2+||v_1||_\alpha^2+|v_2|_\alpha^2)^{p_0}\{(1+||u_1||_\alpha^2+|u_2|_\alpha^2+||v_1||_\alpha^2 +|v_2|_\alpha^2)^{p_1-p_0}\}<\frac{\gamma}{C}\Big\}$$

then we can verify that W is a well posed set of the Problem (2).

In fact, if $(\{u_0,u_1\},\{v_0,v_1\})$ belongs to W it follows by the same as in (c) that $(\{u_k(0),u_k'(0)\},\{v_k(0),v_k'(0)\})$ belongs to W for sufficiently large number $k\geqq k_1$.

Here we have from the equalities of (2)$'$,

$$\begin{cases}(u_k'',u_k')_\alpha+((u_k,u_k'))_\alpha+\gamma_0|u_k'|_\alpha^2+(u_k'f_0(u_k,u_k',v_k,v_k'),u_k')_\alpha=0\\ (v_k'',v_k')_\alpha+((v_k,v_k'))_\alpha+\gamma_1|v_k'|_\alpha^2+(v_k'f_1(u_k,u_k',v_k,v_k'),v_k')_\alpha=0\end{cases}$$
for $t\in(0,\varepsilon_k)$.

Thus by summing these equalities, and using (1, 2, 1), it holds,

$$\frac{d}{dt}(|u_k'|_\alpha^2+|v_k'|_\alpha^2+||u_k||_\alpha^2+||v_k||_\alpha^2) \\ \leqq-\gamma_0|u_k'|_\alpha^2-\gamma_1|v_k'|_\alpha^2+C_0|u_k'|_\alpha^2(|u_k'|_\alpha^2+|v_k'|_\alpha^2+||u_k||_\alpha^2+||v_k||_\alpha^2))^{p_0} \\ +C_1|v_k'|_\alpha^2(|u_k'|_\alpha^2+|v_k'|_\alpha^2+||u_k||_\alpha^2+||v_k||_\alpha^2)^{p_1}$$

$$\leqq (|u_k'|_\alpha^2+|v_k'|_\alpha^2)\{-\gamma+C(|u_k'|_\alpha^2+|v_k'|_\alpha^2+\|u_k\|_\alpha^2+\|v_k\|_\alpha^2)^{p_0}$$
$$\times\{1+(|u_k'|_\alpha^2+|v_k'|_\alpha^2+\|u_k\|_\alpha^2+\|v_k\|_\alpha^2)^{p_1-p_0}\}\}.$$

Therefore for $k\geqq k_1$ $\{u_k(t), v_k(t)\}$ should exist in the interval $[0,\infty)$ from a generalization of Lemma 2 and they satisfy for any $t\in[0,\infty)$,

$$(|u_k'(t)|_\alpha^2+|v_k'(t)|_\alpha^2+\|u_k(t)\|_\alpha^2+\|v_k(t)\|_\alpha^2)^{p_0}\{1+(|u_k'(t)|_\alpha^2$$
$$+|v_k'(t)|_\alpha^2+\|u_k(t)\|_\alpha^2+\|v_k(t)\|_\alpha^2)^{p_1-p_0}\}<\frac{\gamma}{C},$$

and hence it holds that for any $T>0$,

$$\sup_{k\geqq k_1}\sup_{t\in[0,T]}\{|u_k'(t)|_\alpha^2+|v_k'(t)|_\alpha^2+\|u_k(t)\|_\alpha^2+\|v_k(t)\|_\alpha^2\}<\left(\frac{\gamma}{C}\right)^{\frac{1}{p_0}}.$$

Conseqently by utilizing the Theorem 2 we can arrive at the conclusion.

(q. e. d)

Example 1.

$$u''-\Delta u+u'\pm(u')^p=0$$
$$(p\text{: positive integer with } p>1)$$

For the above equation, the well posed set S is given by

$$S=\left\{\{u_0(x),u_1(x)\}\in \mathring{H}^{\alpha+3}(\Omega)\times\mathring{H}^{\alpha+2}(\Omega)\,|\ \|u_0\|_\alpha^2+|u_1|_\alpha^2<\left[\frac{1}{C(p)}\right]^{\frac{2}{p-1}}\right\}$$

where $C(p)$ is the minimum constant which verifies for $u\in\mathring{H}^\alpha(\Omega)$

$$|\langle u^p,(-\Delta)^\alpha u\rangle|\leqq \text{const}\,|u|_\alpha^{p+1}\qquad \text{(from Lemma 1).}$$

Because, apriori we have

$$\langle u'',(-\Delta)^\alpha u'\rangle+\langle -\Delta u,(-\Delta)^\alpha u\rangle+\langle u',(-\Delta)^\alpha u'\rangle$$
$$\pm\langle (u')^p,(-\Delta)^\alpha u'\rangle=0$$

and therefore

$$\frac{d}{dt}\{|u'|_\alpha^2+\|u\|_\alpha^2\}\leqq -|u'|_\alpha^2+C(p)|u'|_\alpha^{p+1}$$
$$=|u'|_\alpha^2\{-1+C(p)|u'|_\alpha^{p-1}\}$$
$$\leqq |u'|_\alpha^2\left\{-1+C(p)(|u'|_\alpha^2+\|u\|_\alpha^2)^{\frac{p-1}{2}}\right\}.$$

From this we know that S is well posed.

Example 2.

$$u''-\Delta u+u'\pm(u)^p(u')^q=0$$
$$(p,q \text{ are positive integers})$$

For the above equation the well posed set S is given by

$$S=\left\{\{u_0(x), u_1(x)\}\in \mathring{H}^{\alpha+3}(\Omega)\times \mathring{H}^{\alpha+2}(\Omega) \mid \|u_0\|_\alpha^2+|u_1|_\alpha^2<\left[\frac{1}{C(p,q)}\right]^{\frac{2}{p+q-1}}\right\}$$

where $C(p,q)$ is a minimum constant which verifies for $u\in\mathring{H}^{\alpha+1}(\Omega)$, $v\in\mathring{H}^{\alpha}(\Omega)$

$$|\langle(u)^p(v)^q, (-\Delta)^\alpha v\rangle|\leqq \text{const}\|u\|_\alpha^p|v|_\alpha^{q+1}.$$

Example 3.

$$\begin{cases} u''-\Delta u+u'(1+u^{p_0}+v^{q_0}+(u')^{r_0}+(v')^{s_0})=0 \\ v''-\Delta v+v'(1+u^{p_1}+v^{q_1}+(u')^{r_1}+(v')^{s_1})=0 \end{cases}$$

(p_i, q_i, r_i, s_i are positive integers)

For the above system of equations we can verify the existence of a well posed set although we leave out the concrete form.

Remark 1. One can easily confirm that these nonlinear functions appeared in the examples satisfy the assumptions $(1,1,1)\sim(1,1,3)$ or $(1,2,1)\sim(1,2,3)$.

Remark 2. Though we treated the equations with the nonlinear functions of u, u', we can also verify the same results obtaned above for the functions of $u, u', \nabla u$ by giving same assumptions.

§ 3. Generalization

In this section we consider the Initial-Boundary Value Problems:

$$(3)\qquad \begin{cases} u''+L(D)u-u'\{\gamma+f(u, Du, \cdots, D^m u, u')\}=0 & \text{in } \Omega\times(0,\infty) \\ u(x,0)=u_0(x),\ u'(x,0)=u_1(x) & \text{in } \Omega \\ D^\beta u|_{\partial\Omega}=0\ (|\beta|\leqq m-1) & \text{for } t\in[0,\infty) \end{cases}$$

$$(4)\qquad \begin{cases} \begin{cases} u''+L(D)u+u'\{\gamma_0+f_0(u, Du, \cdots, D^m u, u', v, Dv, \cdots, D^m v, v')\}=0 \\ v''+L(D)v+v'\{\gamma_1+f_1(u, Du, \cdots, D^m u, u', v, Dv, \cdots, D^m v, v')\}=0 \end{cases} \\ \qquad\qquad \text{in } \Omega\times(0,\infty) \\ \begin{cases} u(x,0)=u_0(x) \\ u'(x,0)=u_1(x) \end{cases} \qquad \begin{cases} v(x,0)=v_0(x) \\ v'(x,0)=v_1(x) \end{cases} \quad \text{in } \Omega \\ D^\beta u|_{\partial\Omega}=D^\beta v|_{\partial\Omega}=0\quad (|\beta|\leqq m-1) \qquad \text{for } t\in[0,\infty). \end{cases}$$

where $L=L(D)=\sum_{|b|\leqq 2m} a_b D^b$ is a uniformly elliptic and coercive operator on $\mathring{H}^m(\Omega)$ with constant coefficients, and f, f_i satisfy same conditions as $(1,1,1)\sim(1,1,3)$ and $(1,2,1)\sim(1,2,3)$ respectively. We shall omit to write out them.

Here we put p as a minimum integer with

$$mp\geqq\left[\frac{n}{2}\right]+1=\alpha.$$

For the Problem (3) we construct approximating solutions $\left\{u_k(t)=\sum_{j=1}^{k}\lambda_{k,j}(t)\psi_j\right\}$ by

$$(3)' \qquad \begin{cases}\langle u_k'', L^p\psi_j\rangle+\langle Lu_k, L^p\psi_j\rangle+\langle u_k'\{\gamma+f(\cdots)\}, L^p\psi_j\rangle=0\\ u_k(0)=\sum_{j=1}^{k}\alpha_j\psi_j,\quad u_k'(0)=\sum_{j=1}^{k}\beta_j\psi_j\end{cases}$$

where $\{\psi_j\}$ are eigen functions of $L^{p+3}(D)$ and

$$u_k(0)\longrightarrow u_0\quad (s)\qquad \text{in}\quad \overset{\circ}{H}{}^{m(p+3)}(\Omega)$$
$$u_k'(0)\longrightarrow u_1\quad (s)\qquad \text{in}\quad \overset{\circ}{H}{}^{m(p+2)}(\Omega).$$

For the Problem (4), we give approximating solutions $\left\{u_k(t)=\sum_{j=1}^{k}\mu_{k,j}(t)\psi_j,\right.$ $\left.v_k(t)=\sum_{j=1}^{k}\nu_{k,j}(t)\psi_j\right\}$ by

$$(4)' \qquad \begin{cases}\begin{cases}\langle u_k'', L^p\psi_j\rangle+\langle Lu_k, L^p\psi_j\rangle+\langle u_k'\{\gamma_0+f_0(\cdots)\}, L^p\psi_j\rangle=0\\ \langle v_k'', L^p\psi_j\rangle+\langle Lv_k, L^p\psi_j\rangle+\langle v_k'\{\gamma_1+f_1(\cdots)\}, L^p\psi_j\rangle=0\end{cases}\\ \qquad\qquad (j=1,2,\cdots,k)\\ u_k(0)=\sum_{j=1}^{k}\alpha_j\psi_j,\quad u_k'(0)=\sum_{j=1}^{k}\beta_j\psi_j,\quad v_k(0)=\sum_{j=1}^{k}d_j\psi_j,\\ v_k'(0)=\sum_{j=1}^{k}e_j\psi_j\end{cases}$$

where

$$u_k(0)\longrightarrow u_0\quad (s)\qquad \text{in}\quad \overset{\circ}{H}{}^{m(p+3)}(\Omega)$$
$$u_k'(0)\longrightarrow u_1\quad (s)\qquad \text{in}\quad \overset{\circ}{H}{}^{m(p+2)}(\Omega)$$
$$v_k(0)\longrightarrow v_0\quad (s)\qquad \text{in}\quad \overset{\circ}{H}{}^{m(p+3)}(\Omega)$$
$$v_k'(0)\longrightarrow v_1\quad (s)\qquad \text{in}\quad \overset{\circ}{H}{}^{m(p+2)}(\Omega)$$

Then we can prove the following theorems by the analogous arguements in the previous section.

Theorem 5.

If the solutions $\{u_k(t)\}$ *of* (3)′ *exist in* $[0, T]$ *independently of* k, *and satisfy*

$$\sup_k \sup_{t\in[0,T]} \{\langle u_k', L^p u_k'\rangle+\langle u_k, L^{p+1}u_k\rangle\}<+\infty$$

then we can find one and only one function which solves the Problem (3).

Theorem 6.

If the solutions $\{u_k(t), v_k(t)\}$ *of* (4)′ *exist in* $[0, T]$ *independently of* k, *and satisfy*

$$\sup_k \sup_{t\in[0,T]} \{(u_k', L^p u_k')+\langle v_k', L^p v_k'\rangle+\langle u_k, L^{p+1}u_k\rangle+\langle v_k, L^{p+1}v_k\rangle\} <+\infty,$$

then we can solve the Problem (4) *uniquely.*

Theorem 7.

① *If the dissipative coefficient* γ *is positive, then we have a non-trivial well posed set of the Problem* (3), *and*

② *if the coefficients* γ_i $(i=0,1)$ *are positive, then we have a non-trivial well posed set of the Problem* (4).

Now in the next, we are concerned with the first order evolution equations. Though the global existence of classical solutions of the Initial-Boundary Value Problems of non-linear parabolic equations are known by the mothod of Green function, giving positive initial values with small norm, we can also give the proofs by the one introduced here giving suitable initial values which are not always positive.

For example, if we consider the problem:

$$(5)\quad \begin{cases} u'-\Delta u+u^p=0 & (p\text{: any positive integer) in } \Omega\times(0,\infty) \\ u(x,0)=u_0 & \text{in } \Omega \\ u|_{\partial\Omega}=0 & t\geqq 0, \end{cases}$$

we can find its well posed set as follows.

Since

$$(u',u)_\alpha+((u,u))_\alpha+(u^p,u)_\alpha=0$$

holds apriori, so it follows that

$$\begin{aligned} \frac{d}{dt}|u|_\alpha^2 &\leqq -\|u\|_\alpha^2+C(p)|u|_\alpha^{p+1} \\ &\leqq -\frac{1}{C_0}|u|_\alpha^2+C(p)|u|_\alpha^{p+1} \\ &=\frac{1}{C_0}|u|_\alpha^2(-1+C_0\cdot C(p)|u|_\alpha^{p-1}), \end{aligned}$$

here the constants $C_0, C(p)$ are minimums which satisfy

$$\begin{aligned} &|u|_\alpha^2\leqq \text{const}\,\|u\|_\alpha^2 \quad \text{for} \quad u\in \mathring{H}^{\alpha+1}(\Omega) \\ &|(u^p,u)_\alpha|\leqq \text{const}\,|u|_\alpha^{p+1} \quad \text{for} \quad u\in \mathring{H}^{\alpha}(\Omega). \end{aligned}$$

Thus a well posed set is given by

$$S=\left\{u_0\in \mathring{H}^{\alpha+3}(\Omega) \mid |u_0|_\alpha<\left[\frac{1}{C_0\cdot C(p)}\right]^{\frac{1}{p-1}}\right\}.$$

Moreover we can find out well posed sets of the problems;

(6) $$\begin{cases} \begin{cases} u' - \Delta u + u^{p_0} + v^{q_0} + u^{r_0} \cdot v^{s_0} = 0 \\ v' - \Delta v + u^{p_1} + v^{q_1} + u^{r_1} \cdot v^{s_1} = 0 \end{cases} \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0 \end{cases}$$

(7) $$\begin{cases} u' + L(D)u + f(u, Du, \cdots, D^{m-1}u) = 0 \\ u(x, 0) = u_0(x) \\ D^\beta u|_{\partial\Omega} = 0 \qquad (|\beta| \leqq m-1). \end{cases}$$

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(Ricevita la 9-an de septembro, 1974)
(Reviziita la 16-an de decembro, 1974)