1. Quite recently, G.R. Sell [6, 7] has shown that there is a way of viewing the solutions of nonautonomous differential equations as dynamical systems. This view point is very general and includes all differential equations satisfying only the weakest hypotheses. In [6], by introducing the concept of the "set of limiting equations" for a given differential equation, he has investigated some of the implications of topological dynamics in this setting. Further, by using the concept of $\omega$-limit set in [7], G.R. Sell has established the basic relationship between the solutions of a given differential equation and the solutions of the corresponding limiting equations. His results generalize some of the earlier works of L. Markus [4] and R.K. Miller [5].

In this paper we introduce concepts of prolongation and prolongational limit sets for the solution of a given differential equation in the usual way and study the properties of these sets. These are then used to discuss the behavior of solutions of limiting equations and to discuss some of the stability properties of the solutions of the given differential equation. The above concepts are well-known in the theory of dynamical systems [1] and its usefulness in stability theory have been established in recent years by many authors including Ura [8] and Bhatia and Hajek [2]. Section 2 deals with definitions and basic lemmas. In Section 3, we obtain some results which relate the behavior of a positively compact solution of the given differential equation with the behavior of solutions of the corresponding limiting equations. This study includes some of the results of G.R. Sell [7]. In Section 4, we have some theorems which characterize the stability properties of solutions of a given differential equation in terms of its prolongation and prolongational limit sets. To motivate the nature of the result contained in Theorem 4.1 we recall the following well-known theorem of T. Ura for dynamical systems defined on locally compact spaces:

**Theorem** (T. Ura, [8]). A compact set $M$ is stable if and only if $D^+(M) = M$.

Further we establish some relationship between the stability of the local dynamical system $\pi$ (see [2] for definition) and the stability of solution of the corresponding differential equation.
2. Let \( W \) be an open set in \( \mathbb{R}^n \), euclidean \( n \)-space. The euclidean norm on \( \mathbb{R}^n \) will be denoted by \(|x|\). Let \( C=C(W\times\mathbb{R}, \mathbb{R}^n) \) denote the set of all continuous functions \( f \) defined on \( W\times\mathbb{R} \) with values in \( \mathbb{R}^n \). We shall say that a function \( f \) is admissible [6] if (i) \( f\in C \), and (ii) the solutions of the differential equation \( x'=f(x, t) \) are unique. By the second condition we mean that given any point \((x_0, t_0)\) in \( W\times\mathbb{R} \), there is precisely one solution \( \phi \) of \( x'=f(x, t) \) that satisfies \( \phi(t_0)=x_0 \). It is evident that if \( f \) is an admissible function, then every translate \( f_\tau \) of \( f \) (where \( f_\tau(x, t)=f(x, \tau+t) \)) is an admissible function. Also, if \( f \) satisfies the global existence property, then so does each \( f_\tau \). Let \( F=(f_\tau: \tau\in\mathbb{R}) \) be the space of translates of \( f \), then \( F \) is a subset of \( C \). Now let \( f \) be an admissible function and consider the space of translates \( F \) in the compact open topology [3] on \( C \). Let \( F^*=\text{Cl} F \) (that is, the closure in the compact open topology) be the hull of \( f \). We shall say that \( f \) is regular [6] if every function \( f^* \) in the hull \( F^* \) is admissible. It is shown in [6] that the mapping \( \pi^*: F^*\times\mathbb{R}\rightarrow F^* \), defined by \( \pi^*(f^*, t)=f^*_t \), is a dynamical system on \( F^* \), when \( F^* \) has compact open topology. Let \( \pi_{f^*} \) denote the flow \( \pi^*(f, t) \) with initial point \( f \). We shall say that the motion \( \pi_{f^*} \) is positively compact [6] if the closure of \( \{\pi^*(f, t) : t\geq 0\} \) lies in a compact subset of \( F^* \) (Negative compactness and compactness are defined similarly).

**Definition of Limiting Equation** [6]. Let \( f\in C \) and let \( F^* \) be the hull of \( f \) (Neither regularity nor admissibility of \( f \) will be important here). Let \( \pi^*(f^*, t)=f^*_t \) be the flow on \( F^* \) and let \( \mathcal{Q}_{f^*} \) denote the \( \omega \)-limit set of \( f \) in this flow. If the \( \omega \)-limit set \( \mathcal{Q}_{f^*} \) of \( f \) in \( F^* \) is nonempty, then we say that the set of limiting equations for

\[
\begin{align*}
(2.1) \quad x' &= f(x, t) \\
(2.2) \quad x' &= f^*(x, t)
\end{align*}
\]

is the set of all differential equations of the form

where \( f^*\in\mathcal{Q}_{f^*} \).

**Standing Hypothesis**: Throughout the paper we shall consider only those differential equations (2.1) for which \( f \) does not belong to either the positive limit set \( \mathcal{Q}_{f^*} \) or the negative limit set \( \mathcal{A}_{f^*} \) of the motion \( \pi_{f^*} \). This is justified as our interest is to compare the behavior of solutions of (2.1) with the corresponding behavior of solutions of (2.2). This would imply that \( F \) is an open set in \( F^* \), as is easily proved.

Let \( f\in C \) be an admissible function and let \( \phi(t)=\phi(x, f, t) \) be the solution of (2.1) that satisfies \( \phi(x, f, 0)=x \). Let \( I_{(x, f)}=(\alpha, \beta) \) be the maximal interval of definition of \( \phi \). We shall say that the solution \( \phi \) is positively compact if the set \( \{\phi(t) : 0\leq t<\beta\} \) lies in a compact set in \( W \) (Negative compactness and compactness are defined similarly). Since \( I_{(x, f)} \) is maximal, it follows that \( \beta=+ \)
$\infty$, whenever $\phi$ is positively compact. If $\phi$ is compact, then $I(x, f)$ is $R$. This
of $\phi$ instead of $\phi$ is defined for all $t \in R$.

Let $f \in C$ be a regular function. Then the mapping [6, Theorem 8]
\[ \pi(x, f^*; t) = (\phi(x, f^*; t), f^*) \]
defines a local dynamical system on $W \times F^*$, where $F^*$ is the hull of $f$. It is
clear that the motion $\pi(x, f^*; t)$ is defined for all $t \geq 0$ if and only if the solution
$\phi(x, f^*, t)$ is defined for all $t \geq 0$.

For $(x, f^*) \in W \times F^*$, $\tau(x, f^*) = [(\pi(x, f^*; t) : t \in R)$ is the trajectory through
$(x, f^*)$. Similarly $\tau^+(x, f^*) = [(\pi(x, f^*; t) : t \geq 0)$ and $\tau^-(x, f^*) = [(\pi(x, f^*; t) :
t \leq 0)$ are defined to be, respectively, the positive and negative semitrajectory
through $(x, f^*)$.

The limit sets, prolongations, and prolongational limit sets may now be defined as follows.

**Definition 2.1.** For each $(x, f^*) \in W \times F^*$, let
\[ \Omega(x, f^*) = \{ (y, g) \in W \times F^* : \pi(x, f^*; t) \rightarrow (y, g) \}
\]
and $D^+(x, f^*) = \{ (y, g) \in W \times F^* : \pi(x, f^*; t) \rightarrow (y, g) \}$ for some sequence $t \rightarrow +\infty$
and $(x^t, f^t) \in W \times F^*$ such that $\pi(x^t, f^t) \rightarrow (x, f^*)$, and
\[ J^+(x, f^*) = \{ (y, g) \in W \times F^* : \pi(x^t, f^t; t^t) \rightarrow (y, g) \}
\]
for some sequences $t^t \rightarrow +\infty$ and $(x^t, f^t) \in W \times F^*$ such that $(x^t, f^t) \rightarrow (x, f^*)$.

$\Omega(x, f^*)$ is called the positive limit set, $D^+(x, f^*)$ the positive prolongation, and
$J^+(x, f^*)$ the positive prolongational limit set of $(x, f^*)$. The negative limit set
$A^+(x, f^*)$, negative prolongation $D^-(x, f^*)$, and negative prolongational
limit set $J^-(x, f^*)$ are defined similarly.

Now consider the projection mapping $P : W \times F^* \rightarrow W$.

**Definition 2.2.** Define
\[ \Omega_{r}(x, f^*) = P(\Omega(x, f^*)), D_{r}^+(x, f^*) = P(D^+(x, f^*)), \text{ and } J_{r}^+(x, f^*) = P(J^+(x, f^*)).
\]
$\Omega_r(x, f^*)$ is called the positive extended limit set, $D_r^+(x, f^*)$ the positive extended
prolongational limit set of the solution $\phi(x, f^*, t)$.

**Remark 2.1.** It is clear that for any $(x, f^*) \in W \times F^*$, $D_{r}^+(x, f^*) = \tau_{r}^+(x, f^*)$ and $J_{r}^+(x, f^*) \subseteq \tau_{r}^+(x, f^*)$ and $J_{r}^+(x, f^*) \subseteq \tau_{r}^+(x, f^*)$, where $\tau_{r}^+(x, f^*) = P(\tau^+(x, f^*))$.

**Remark 2.2.** For $(y, g) \in J^+(x, f)$, then $g \in \Omega_f$. For, by definition of $J^+(x, f)$, there are sequences $f^t \in F^*, f^t \rightarrow f$ and $t^t \rightarrow +\infty$ such that $\pi^+(f^t, t^t) \rightarrow g$. But we can assume that $f^t = \pi^+(f, t^t)$ for some sequence $(t^t)$ in $R$, as $F$ is an
open set (recall the remark explained in Standing Hypothesis above). Obviously $T^t \rightarrow 0$, otherwise it gives a contradiction to the fact that $f \in \Omega_f \cup A_f$. This
implies that $\pi^+(f^t, t^t) = \pi^+(f, T^t + t^t)$ and $T^t + t^t \rightarrow +\infty$. Hence $g \in \Omega_f$.

It is proved in [7] that if the motion $\pi_f$ in $F^*$ is positively compact, then
$\Omega_{r}(x, f)$ can be characterized as follows.

**Lemma 2.1.** [7, Lemma 2]. Let $f \in C$ be a regular function and assume
that the motion $\pi_f^*$ is positively compact. Then a point $\hat{x}$ lies in $\Omega_\delta(x,f)$ if and only if there is a sequence $(\tau^n)$ in $R$ with $\tau^n \to +\infty$ and $\phi(x,f,\tau^n) \to \hat{x}$.

Similar characterizations for $J_f^+(x,f)$ and $D_f^+(x,f)$ are given in the following two lemmas.

**Lemma 2.2.** Let $f \in C$ be a regular function and assume that the motion $\pi_f^*$ is positively compact. Then a point $\hat{x}$ lies in $J_f^+(x,f)$ if and only if $\phi(x^n,f,\tau^n) \to \hat{x}$ for some sequences $x^n \to x$ and $\tau^n \to +\infty$.

**Proof.** Let $\hat{x} \in J_f^+(x,f)$. By the definition of $J_f^+(x,f)$, there is an $\hat{f}$ in $F^*$ such that $(\hat{x},\hat{f}) \in J^+(x,f)$. Further, there are sequences $(y^n,g^n)$ and $(\tau^n)$ such that $(y^n,g^n) \to (x,f)$, $\tau^n \to +\infty$, and $\pi(y^n,g^n;\tau^n) \to (\hat{x},\hat{f})$. This implies that there are sequences $(y^n)$, $(g^n)$ and $(\tau^n)$ such that $y^n \to x$, $g^n \to f$, $\tau^n \to +\infty$ and $\phi(y^n,g^n,\tau^n) \to \hat{x}$. Now we can assume that the sequence $(g^n)$ is of the form $(\pi^*(f,T^n))$, where $T^n \to 0$ (see Remark 2.2). Let $\phi(y^n,\pi^*(f,T^n),-T^n) = x^n$. As $y^n \to x$ and $T^n \to 0$, we have $x^n \to x$ and $\phi(x^n,f,t^n + T^n) \to \hat{x}$. Hence, if $\hat{x} \in J_f^+(x,f)$ then there are sequences $x^n \to x$ and $\tau^n = t^n + T^n \to +\infty$ such that $\phi(x^n,f,t^n) \to \hat{x}$. Conversely, suppose that $\phi(x^n,f,\tau^n) \to \hat{x}$ for some sequences $x^n \to x$ and $\tau^n \to +\infty$. Consider the sequence $(\pi^*(f,\tau^n))$. Since $\pi^*_f$ is positively compact, there exists a convergent subsequence $(\pi^*(f,\tau^{n_k}))$ of $(\pi^*(f,\tau^n))$. Let $\pi^*(f,\tau^{n_k}) \to \hat{f} \in F^*$. Thus, we have $x^{n_k} \to x$, $\tau^{n_k} \to +\infty$, $\phi(x^{n_k},f,\tau^{n_k}) \to \hat{x}$ and $\pi^*(f,\tau^{n_k}) \to \hat{f}$. This implies that $\pi(x^{n_k},f,\tau^{n_k}) \to (\hat{x},\hat{f})$ for some sequences $x^{n_k} \to x$ and $\tau^{n_k} \to +\infty$. Therefore $(\hat{x},\hat{f}) \in J^+(x,f)$, which implies that $\hat{x} \in J_f^+(x,f)$. This completes the proof.

**Lemma 2.3.** Let $f \in C$ be a regular function and assume that the motion $\pi_f^*$ is positively compact. Then a point $\hat{x}$ lies in $D_f^+(x,f)$ if and only if $\phi(x^n,f,\tau^n) \to \hat{x}$ for some sequences $x^n \to x$, $\tau^n \geq 0$.

**Proof.** If $\hat{x} \in D_f^+(x,f)$, then, by Remark 2.1, either $\hat{x} \in \Omega_\delta^+(x,f)$ or $\hat{x} \in J_f^+(x,f)$. If $\hat{x} \in \Omega_\delta^+(x,f)$, then there is a $\tau \geq 0$ such that $\phi(x,f,\tau) = \hat{x}$ and hence $\phi(x^n,f,\tau^n) \to \hat{x}$ for the constant sequences $x^n = x$ and $\tau^n = \tau$. The rest of the proof is similar to that of Lemma 2.2 and hence omitted.

3. Now we shall give some theorems which relate the compactness property of solutions of (2.1) with the corresponding behavior of solutions of (2.2).

**Theorem 3.1.** Let $f \in C$ be a regular function and assume that the motion $\pi_f^*$ is compact. Let $x \in W$ and $N$ be a neighborhood of $x$ such that $\text{Cl} \bigcup_{r \in R} \{\phi(y,f_r,t) : y \in N, t \geq 0\}$ is a compact subset of $W$. Then the sets $D^+(x,f)$ and $J^+(x,f)$ are nonempty, compact and connected. If $(x^*,f^*) \in J^+(x,f)$, then the solution $\phi(x^*,f^*,t)$ of (2.2) is compact. Furthermore, $D^+_f(x,f)$ and $J^+_f(x,f)$ are nonempty, compact and connected.

**Proof.** Let $N$ be a neighborhood of $x$ such that $\text{Cl} \bigcup_{r \in R} \{\phi(y,f_r,t) : y \in N, t \geq 0\}$ is a compact subset of $W$. If $\phi(x,f,\tau) = \hat{x}$ for some $\tau \geq 0$, then there is a $\tau_0 \geq 0$ such that $\phi(x,f,\tau) \in N$, for all $\tau \geq \tau_0$. This implies that $\phi(x,f,\tau) \to \hat{x}$ for some sequence $\tau \to +\infty$. Since $\phi(x^n,f,\tau^n) \to \hat{x}$ for some sequences $\tau^n \to +\infty$, it follows that $\phi(x^n,f,\tau^n) \to \hat{x}$ for some sequences $x^n \to x$ and $\tau^n \to +\infty$. Therefore, $\hat{x} \in D_f^+(x,f)$ and $\hat{x} \in J_f^+(x,f)$. This completes the proof.
$\geq 0$] be a compact subset of $W$. Consider the set $N \times F$, where $F = \{f_r : r \in \mathbb{R}\}$. Obviously $N \times F$ is a neighborhood of $(x, f)$. Further, we have $\text{Cl}[\pi(N \times F; R')] \subset \text{Cl}[ \bigcup_{r \in \mathbb{R}} \{f(y, f_r, t) : y \in N, t \geq 0\} \times F^*$, where $\pi(N \times F; R')$ denotes the set $(\pi(x, f_r, t), (x, f_r) \in N \times F, t \geq 0)$. Since the motion $\pi_f^*$ is compact, $F^*$ is compact and hence $\text{Cl}[\pi(N \times F; R')]$ is compact. Now the application of Theorem 6.7 [2, p. 63] yields that the sets $D^+(x, f)$ and $J^+(x, f)$ are nonempty, compact and connected. The rest of the proof follows from the fact that the projection mapping $P : W \times F^* \rightarrow W$ is continuous.

**Remark 3.1.** G.R. Sell [7, Theorem 1] assumed that the solution $\phi(x, f, t)$ of (2.1) is positively compact to prove a similar result for the $\omega$-limit set $\mathcal{O}_x$ $(x, f)$. The following example shows that the assumption of positive compactness of the solution $\phi(x, f, t)$ is not sufficient to conclude Theorem 3.1.

**Example 3.1.** Consider the differential system
\[
\dot{x}_1 = -x_1/(t+1), \quad \dot{x}_2 = x_2/(t+1), \quad t \geq 0.
\]
It is easy to show that for any $x = (a, 0) \in \mathbb{R}^2$, where $a \neq 0$, $\phi(x, f, t)$ is positively compact but $J_1^+(x, f) = \{(x_1, x_2) : x_1 = 0, x_2 \in \mathbb{R}\}$ is not compact.

**Theorem 3.2.** Let $f \in C$ be a regular function and assume that the motion $\pi_f^*$ is positively compact. Let $D^+(x, f)$ be a compact subset of $W \times F^*$. Then for every point $(x^*, f^*) \in J^+(x, f)$, the solution $\phi(x^*, f^*, t)$ of (2.2) is compact. Moreover, there exist sequences $\{x^i\}$ in $W$ and $\{\tau^i\}$ in $R$ with $x^i \rightarrow x$, $\tau^i \rightarrow +\infty$ such that $\phi(x^i, f, \tau^i + t)$ converges to $\phi(x^*, f^*, t)$ uniformly on compact sets in $R$. 

**Proof.** The first part follows from Theorem 3.1. For the second part, since $\phi(x^i, f^*, t)$ is compact it follows that $I_{x^i, f^*} = R$. Therefore, the solution $\phi(x^*, f^*, t)$ and the motion $\pi(x^*, f^*; t)$ are defined for all $t$ in $R$. Now let $\{x^i\}$ and $\{\tau^i\}$ be sequences in $W$ and $R$ respectively, with $x^i \rightarrow x$, $\tau^i \rightarrow +\infty$ and $\pi(x^i, f^*; \tau^i) \rightarrow (x^*, f^*)$. Since $\pi$ is continuous $\pi(x^i, f; \tau^i + t)$ converges to $\pi(x^*, f^*; t)$ uniformly on compact sets in $R$ [6, Lemma 4]. Further, since the projection mapping $P$ is continuous, the solutions $\phi(x^i, f, \tau^i + t) = P(\pi(x^i, f, \tau^i + t))$ converge to $\phi(x^*, f^*, t)$ uniformly on compact sets in $R$. This completes the proof.

The conclusion of Theorem 3.2 can be formulated in terms of the positive prolongational limit set $J^+_{x^*}(x, f)$. The basic fact here is that if $x^* \in J^+_{x^*}(x, f)$, then there is an $f^* \in F^*$ such that $(x^*, f^*) \in J^+(x, f)$.

**Corollary 3.1.** Let $f$, $\pi_f^*$, and $D^+(x, f)$ be as in the preceding theorem. Then for every point $x^* \in J^+_{x^*}(x, f)$, there is a function $f^* \in \Omega_f^*$ such that the solution $\phi(x^*, f^*, t)$ of (2.2) is compact.

This corollary asserts that the positive prolongational limit set $J^+_{x^*}(x, f)$ is quasi-invariant in the sense defined by R.K. Miller [5].
If $f$ is asymptotically autonomous [7], that is, if $\mathcal{Q}_f^*$ is a singleton, then one can say more.

**Corollary 3.2.** Let $f \in C$ be a regular function that is asymptotically autonomous. Let $\mathcal{Q}_f^* = \{f^*\}$. Then every positive prolongational limit set $J^+(x, f)$ can be expressed in the form

$$J^+(x, f) = J_{\phi^*}^+(x, f) \times \{f^*\}. $$

Therefore, $J_{\phi^*}^+(x, f)$ is the union of solutions of $x' = f^*(x)$. If $D^+(x, f)$ is compact, then for every $x^*$ in $J_{\phi^*}^+(x, f)$, the solution $\phi(x^*, f^*, t)$ is compact.

**Theorem 3.3.** Let $f \in C$ be regular function and assume that the motion $\pi_f^*$ is positively compact. Let $\mathcal{Q}_f(x, f)$ be a nonempty compact subset of $W$. Then the solution $\phi(x, f, t)$ of (2.1) is positively compact.

**Proof.** Consider the set $\mathcal{Q}(x, f)$. Since $\mathcal{Q}(x, f) \subset \mathcal{Q}_f(x, f) \times F^*$ and $\mathcal{Q}(x, f)$ is closed, obviously the set $\mathcal{Q}(x, f)$ is compact. This implies that the closure of $\tau^+(x, f)$ is compact [2, Lemma 5.8]. Further, since the projection mapping $P$ is continuous, it follows that the solution $\phi(x, f, t)$ of (2.1) is positively compact.

**Theorem 3.4.** Let $f \in C$ be a regular function and assume that the motion $\pi_f^*$ is positively compact. Then $\mathcal{Q}_f(x, f)$ is nonempty whenever $J_{\phi^*}^+(x, f)$ is nonempty and compact.

The proof is direct and simple and hence omitted.

4. In this section we shall give some results which characterize the stability property of solutions of (2.1) in terms of its prolongation and prolongational limit sets.

**Lemma 4.1.** Let $f \in C$ be a regular function and assume that the motion $\pi_f^*$ is positively compact. If the solution $\phi(x, f, s)$ of (2.1) is stable, then

$$D_{\phi^*}^+(x, f) = \text{Cl}(\gamma_{\phi^*}^+(x, f)).$$

**Proof.** We know that $D_{\phi^*}^+(x, f) = \gamma_{\phi^*}^+(x, f) \cup J_{\phi^*}^+(x, f)$ and $\text{Cl}(\gamma_{\phi^*}^+(x, f)) = \gamma_{\phi^*}^+(x, f) \cup \mathcal{Q}_f(x, f)$. Therefore, we only need to prove that $J_{\phi^*}^+(x, f) = \mathcal{Q}_f(x, f)$. Let $y \in J_{\phi^*}^+(x, f)$. Then, by Lemma 2.2, there exist sequences $x^n \rightarrow x$ and $\tau^n \rightarrow +\infty$ such that $\phi(x^n, f, \tau^n) \rightarrow y$. Let $\varepsilon > 0$ be given. Since $\phi(x, f, t)$ is stable, for given $\varepsilon/2$ there exists a $\delta = \delta(\varepsilon/2) > 0$ such that

$$|\phi(x, f, s) - \phi(x^n, f, \tau^n)| < \varepsilon/2$$

for all $t \geq 0$, whenever $|x - y| < \delta$. As $x^n \rightarrow x$ there is an $N_1 \geq 0$ such that $|x - x^n| < \delta$ for $n \geq N_1$. Thus, we have

$$|\phi(x, f, \tau^n) - \phi(x^n, f, \tau^n)| < \varepsilon/2$$

for $n \geq N_1$. Further, since $\phi(x^n, f, \tau^n) \rightarrow y$, given $\varepsilon/2$ there exists a number $N_2 \geq 0$ such that

$$|\phi(x^n, f, \tau^n) - y| < \varepsilon/2$$

for $n \geq N_2$. Choose $N = \max(N_1, N_2)$. From (4.1) and (4.2), we have
\begin{align*}
|\phi(x, f, \tau^n) - y| \leq |\phi(x, f, \tau^n) - \phi(x^n, f, \tau^n)| + |\phi(x^n, f, \tau^n) - y| < \varepsilon / 2 + \varepsilon / 2
= \varepsilon \quad \text{for} \quad n \geq N.
\end{align*}

This shows that \( \phi(x, f, \tau^n) \to y \). Therefore \( y \in \Omega_\phi(x, f) \). Hence \( D_\phi^+(x, f) = \text{Cl} [\gamma_\phi^+(x, f)] \). This completes the proof.

**Theorem 4.1.** Let \( f \in C \) be a regular function with \( f(0, t) = 0 \) for all \( t \geq 0 \) and assume that the motion \( \pi_{f^*} \) is positively compact. Then the null solution of (2.1) is stable if and only if \( D_\phi^+(0, f) = \{0\} \).

The necessity part of the proof follows from Lemma 4.1 and sufficiency part is direct and hence omitted.

**Corollary 4.1.** Let \( f \in C \) be a regular function with \( f(0, t) = 0 \) for all \( t \geq 0 \) and assume that the motion \( \pi_{f^*} \) is positively compact. Then the null solution of (2.1) is stable if the local dynamical system \( \pi \) on \( W \times F^* \) is stable.

**Proof.** The stability of \( \pi \) implies that for each \( (x, f) \in W \times F^*, \ D^+(x, f) = \text{Cl} [\gamma^+(x, f)] \) [2, Def. 7.7 and Theorem 7.8]. Therefore \( D^+(0, f) = \text{Cl} [\gamma^+(0, f)] \) and by taking the projection \( P \) of both the sets on \( W \), we have \( D_\phi^+(0, f) = \{0\} \). Thus the application of Theorem 4.1 yields the desired result.

**Theorem 4.2.** Let \( f \in C \) be a regular function with \( f(0, t) = 0 \) \( t \geq 0 \) and assume that the motion \( \pi_{f^*} \) is positively compact. Then the null solution of (2.1) is asymptotically stable if and only if \( \{x_0 : J_\phi^+(x_0, f) = \{0\}\} \) is a neighborhood of the origin.

**Proof.** Let the null solution of (2.1) be asymptotically stable. Hence it is stable. Therefore, by Theorem 4.1, we have \( D_\phi^+(0, f) = \{0\} \). Further, for a given \( \delta_0 > 0 \) and for each \( \eta > 0 \) there exists a \( T = T(\eta) > 0 \) such that \( |x_0| < \delta_0 \) implies that \( |\phi(x_0, f, t)| < \eta \) for all \( t \geq T \). This implies that \( J_\phi^+(x_0, f) = \{0\} \), because by taking sequences \( x^n \to x_0 \) and \( \tau^n \to + \infty \), one can easily prove that \( \phi(x^n, f, \tau^n) \to 0 \). Hence \( \{x_0 : |x_0| < \delta_0\} \) is the neighborhood of the origin such that \( J_\phi^+(x_0, f) = \{0\} \). Conversely, suppose that \( \{x_0 : J_\phi^+(x_0, f) = \{0\}\} \) is a neighborhood of the origin. This shows that \( J_\phi^+(0, f) = \{0\} \). Further, we have \( D_\phi^+(0, f) = \gamma_\phi^+(0, f) \cup J_\phi^+(0, f) = \{0\} \). Hence by Theorem 4.1, it follows that the null solution of (2.1) is stable. For the rest of the proof, let \( \{x_0 : |x_0| < \delta_0\} \subset \{x_0 : J_\phi^+(x_0, f) = \{0\}\} \). Then since for each \( x_0 \) satisfying \( |x_0| < \delta_0, J_\phi^+(x_0, f) \) is a nonempty compact set, we have by Theorem 3.4 that the set \( \Omega_\phi(x_0, f) \) is nonempty. Therefore \( \Omega_\phi(x_0, f) = \{0\} \) whenever \( |x_0| < \delta_0 \). This implies that for a given \( \eta > 0 \) there exists a \( T = T(\eta) \) such that whenever \( |x_0| < \delta_0, |\phi(x_0, f, t)| < \eta \) for all \( t \geq T \). This completes the proof.

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