Favard’s Separation Theorem in Functional Differential Equations

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1. Introduction.
For the existence of an almost periodic solution of ordinary differential equations, several kinds of the separation conditions are assumed by many authors [1], [2], [3], [4]. Among them Favard [1] has shown that a linear system
\[ \dot{x} = A(t)x + f(t), \]
where \( A(t) \) and \( f(t) \) are almost periodic, has an almost periodic solution if it has a bounded solution and if for every \( B(t) \in H(A) \), every nontrivial solution \( x(t) \) of
\[ \dot{x} = B(t)x \]
satisfies the condition
\[ \inf_{t \in \mathbb{R}} |x(t)| > 0, \]
whenever \( x(t) \) is defined and bounded on \( \mathbb{R} \) (shortly, \( \mathbb{R} \)-bounded). On the other hand, the famous Amerio’s separation condition [2], which guarantees the existence of an almost periodic solution of an almost periodic system with a bounded solution
\[ \dot{x} = f(t, x), \]
means that for every \( g(t, x) \in H(f) \) every pair of distinct \( \mathbb{R} \)-bounded solutions \( x^1(t) \) and \( x^2(t) \) of
\[ \dot{x} = g(t, x) \]
satisfies the condition
\[ \inf_{t \in \mathbb{R}} |x^1(t) - x^2(t)| \geq \delta > 0 \]
for a constant \( \delta \) independent of the pair of solutions.
As easily can be seen, there is no difficulty in extending these theorems to the case of functional differential equations. Moreover, in Amerio’s theorem, the condition (2) can be obviously replaced by the condition
\[ \inf_{t \in \mathbb{R}} \|x^1 - x^2\| \geq \delta > 0, \]
where $x_t$ denotes the element of $C([-h, 0], R^n)$ defined by $x_t(s) = x(t+s)$ and $\|\phi\| = \sup \{|\phi(s)|; s \in [-h,0]\}$ ($h$ is the lag). However, in Favard's theorem, the replacement of the condition (1) by the condition

$$\inf_{t \in R} \|x_t\| > 0$$

is not obvious. The condition (3) is more desirable than (1). For, as will be shown later, every nontrivial $R$-bounded solution of autonomous linear system satisfies the condition (3), but the solution $x(t) = \sin t$ of

$$\dot{x}(t) = -x\left(t-\frac{\pi}{2}\right)$$

does not satisfy the condition (1).

Our purpose is to prove the following theorem.

**Theorem.** Suppose that $A(t, \phi)$ is continuous in $(t, \phi) \in R \times C([-h, 0], R^n)$ linear in $\phi$ and almost periodic in $t$ uniformly with respect to $\phi$.

If for every $B(t, \phi) \in H(A)$ every nontrivial $R$-bounded solution of the system

$$\dot{x}(t) = B(t, x_t)$$

satisfies the condition (3), then for any almost periodic function $f(t)$ the system

$$\dot{x}(t) = A(t, x_t) + f(t)$$

has an almost periodic solution, whenever it has an $R^+$-bounded solution (namely, a solution which is defined and bounded on $[0, \infty)$).

2. $R$-bounded solution of the system in the hull.

First of all, recall that a continuous function $f(t, \phi)$ is said to be almost periodic in $t$ uniformly with respect to $\phi \in C([-h, 0], R^n)$ if and only if for any sequence $\{t_h\}$ the sequence $\{f(t+t_h, \phi)\}$ contains a subsequence which converges uniformly on $R \times K$ for any compact set $K \subset C([-h,0], R^n)$, where the collection of the limiting functions is the hull $H(f)$ of $f(t, \phi)$.

Furthermore, we know that for any $g(t, \phi) \in H(f)$, we can select a sequence $\{t_h\}, t_h \to \infty$, so that $f(t+t_h, \phi)$ converges to $g(t, \phi)$, see [5].

Noting this fact, we can prove the following lemma by the standard arguments.

**Lemma 1.** Suppose that the continuous function $f(t, \phi)$ is almost periodic in $t$ uniformly with respect to $\phi$ and satisfies the condition

$$\sup \{|f(t, \phi)|; t \in R, \|\phi\| \leq \alpha\} \leq F(\alpha) < \infty$$

for every $\alpha > 0.$
If the system
\begin{equation}
\dot{x}(t) = f(t, x_t)
\end{equation}
has an $R^+$-bounded solution $x(t)$, then for any $g(t, \phi) \in H(f)$ the system
\begin{equation}
\dot{x}(t) = g(t, x_t)
\end{equation}
has an $R$-bounded solution. More exactly, if $\{x(t+t_k), f(t+t_k, \phi)\}$ converges to $(y(t), g(t, \phi))$, then $y(t)$ is a bounded solution of (8) on $(-\lim_{k \to \infty} t_k, \infty)$.

The following lemma can be proved by slightly modifying the proof of Lemma 1 in [6].

**Lemma 2.** Let $f(t, \phi) = A(t, \phi)$ be the one as in Theorem, and then it satisfies the condition (6) with $F(\alpha) = L\alpha$ for a constant $L$.

3. **Norm $||\cdot||_\ast$.**

Now we shall introduce a new norm $||\cdot||_\ast$ in the space $C([-h, 0], R^n)$ defined by

\[ ||\phi||_\ast = \left( \int_{-h}^{0} |\phi(s)|^2 ds \right)^{1/2}, \]

where $|\cdot|$ is the Euclidean norm in $R^n$. Clearly we have

\begin{equation}
||\phi||_\ast \leq \sqrt{h} ||\phi||.
\end{equation}

Conversely if $\phi(s)$ satisfies a Lipschitz condition

\begin{equation}
|\phi(s) - \phi(s')| \leq L |s-s'|
\end{equation}

with the Lipschitz constant $L > 0$, then we have

\begin{equation}
||\phi||_\ast \geq \sqrt{\frac{||\phi||^2}{3L^* ||\phi||}}, \quad L^*(\alpha) = \max \left\{ \frac{1}{h}, \frac{L}{\alpha} \right\}.
\end{equation}

In fact, by choosing $\sigma \in [-h, 0]$ so that $|\phi(\sigma)| = ||\phi||$, we have

\[ |\phi(s)| \geq ||\phi|| - L |s-\sigma|, \]

which implies

\[ ||\phi||_\ast^2 = \int_{-h}^{0} |\phi(s)|^2 ds \geq I(\sigma), \]

where

\[ I(\sigma) = \int_{-h}^{0} [\max \{0, ||\phi|| - L |s-\sigma|\}]^2 ds. \]

Easily we can verify
\[ I(\sigma) \geq I(-h) = \int_0^\sigma (||\varphi|| - Ls)^2 ds \]

for any \( \sigma \in [-h, 0] \) and \( \alpha = \min \left\{ h, \frac{||\varphi||}{L} \right\} \), and hence the inequality (11) follows from the fact (11)

\[ I(-h) = \frac{\alpha^2}{3} ||\varphi||^2 + \frac{\alpha}{3} (2 ||\varphi|| - L\alpha)(||\varphi|| - L\alpha) \geq \frac{\alpha}{3} ||\varphi||^2. \]

**Lemma 3.** Suppose that \( f(t, \phi) \) satisfies the conditions in Lemma 1 and that the system (7) has an R-bounded solution.

Put

\[ \lambda(x) = \sup \{ ||x_t||_*; t \in \mathbb{R} \} \]

for an R-bounded function \( x(t) \), and put

\[ \mathcal{A}(f) = \inf \{ \lambda(x); x(t) \text{ is an R-bounded solution of (7)} \}. \]

Then for every \( g(t, \phi) \in H(f) \), we have \( \mathcal{A}(g) = \mathcal{A}(f) \).

**Proof.** First of all, we note that \( \lambda(x) < \infty \) if \( x(t) \) is R-bounded. For every \( \varepsilon > 0 \), there exists an R-bounded solution \( x(t) \) of (7) such that \( \lambda(x) \leq \mathcal{A}(f) + \varepsilon \). By Lemma 1, for every \( g(t, \phi) \in H(f) \), the system (8) has an R-bounded solution \( y(t) \) to which \( \{x(t+t_k)\} \) converges uniformly on any compact interval for some sequence \( \{t_k\} \). Since

\[ ||y_t||_* - ||x_{t+t_k}||_* \leq ||x_{t+t_k} - y_t||_* \leq \sqrt{h} ||x_{t+t_k} - y_t|| \to 0 \]

by (9), we have

\[ \mathcal{A}(g) \leq \lambda(y) \leq \lambda(x) \leq \mathcal{A}(f) + \varepsilon. \]

This implies \( \mathcal{A}(g) \leq \mathcal{A}(f) \). On the other hand, \( g(t, \phi) \in H(f) \) is almost periodic in \( t \) uniformly with respect to \( \phi \) and \( f(t, \phi) \in H(g) \), and hence \( \mathcal{A}(f) \leq \mathcal{A}(g) \).

Thus, we have \( \mathcal{A}(g) = \mathcal{A}(f) \) for every \( g(t, \phi) \in H(f) \).

**Remark.** More specially, the above proof shows that if \( x(t) \) is R-bounded and \( \{x(t+t_k)\} \) converges to a \( y(t) \) for a \( t_k \) then \( y(t) \) is R-bounded and \( \lambda(y) \leq \lambda(x) \).

**Lemma 4.** Suppose that \( f(t, \phi) \) satisfies the condition (6) with \( F(\alpha) = o(\alpha^3) \) as \( \alpha \to \infty \).

Then, for any \( \alpha > 0 \) there exists a constant \( \beta > 0 \) such that if \( x(t) \) is an R-bounded solution of the system (7) and satisfies \( \lambda(x) \leq \alpha \), we have

\[ ||x_t|| \leq \beta \text{ for all } t \in \mathbb{R}. \]

**Proof.** By the condition \( F(\alpha) = o(\alpha^3) \), we can find a \( \beta > \alpha \sqrt{\frac{3}{h}} \) so that
\[ \frac{F(\tau)}{\tau^3} < \frac{1}{3\alpha^2} \quad \text{for all } \tau \geq \beta. \]

Now we shall show that this \( \beta \) is the desirable one.

Suppose that there exists an \( R \)-bounded solution \( x(t) \) of (7) such that \( \lambda(x) \leq \alpha \) and \( \tau = \sup \{ ||x_\ell||; t \in R \} > \beta \). Clearly, we can find a \( \delta > 0 \) and a \( \tau \in R \) so that
\[ \tau - \delta \geq \beta, \quad \frac{F(\tau)}{(\tau - \delta)^3} < \frac{1}{3\alpha^2} \quad \text{and} \quad |x(\tau)| \geq \tau - \delta. \]

Since \( |x(t)| \leq |f(t, x_t)| \leq F(\tau), x_t \) satisfies (10) with \( L = F(\tau) \). Therefore, from (11) it follows that
\[ \alpha \geq \lambda(x) \geq ||x_\tau|| \geq \sqrt{\frac{||x_\tau||^2}{3L_1}} \geq \sqrt{(\tau - \delta)^2}, \]

\[ L_1 = \max \left\{ \frac{1}{h}, \frac{F(\tau)}{\tau - \delta} \right\}, \]

that is,
\[ \frac{F(\tau)}{(\tau - \delta)^3} \geq \frac{1}{3\alpha^2} \quad \text{or} \quad \frac{3\alpha^2}{h} \geq (\tau - \delta)^2 \geq \beta^2. \]

This contradiction shows that \( ||x_t|| \leq \beta \) for all \( t \in R \) if \( \lambda(x) \leq \alpha \).

Now we shall show the existence of a solution with the minimal norm.

**Lemma 5.** Under the assumptions in Lemma 4, suppose that the system (7) has an \( R \)-bounded solution.

Then there exists an \( R \)-bounded solution \( x(t) \) of (7) with the property \( \lambda(x) = \mathcal{A}(f) \).

**Proof.** By the definition of \( \mathcal{A}(f) \), there exists a sequence \( \{x^k(t)\} \) of \( R \)-bounded solutions of (7) such that \( \lambda(x^k) \leq \mathcal{A}(f) + \frac{1}{k} \leq \mathcal{A}(f) + 1 \). By Lemma 4, there exists a \( \beta > 0 \) such that
\[ ||x_t^k|| \leq \beta \] for all \( k \) and all \( t \in R \).

Hence, \( x_t^k \in K \) for some compact set \( K \subset C([-h, 0], R^n) \), all \( k \) and all \( t \in R \), because \( |x^k(t)| \leq F(\beta) \) for all \( k \) and all \( t \in R \). Thus, \( \{x^k(t)\} \) has a subsequence \( \{x^{k_j}(t)\} \) which converges to an \( R \)-bounded solution \( x(t) \) of (7).

On the other hand, by using the same relation as (12), we have
\[ ||x_t|| \leq ||x_t^k|| + \sqrt{h} ||x_t^k - x_t|| \]
\[ \leq \mathcal{A}(f) + \frac{1}{k_j} + \sqrt{h} ||x_t^j - x_t||, \]

which implies that \( \lambda(x) \leq \mathcal{A}(f) \), that is, \( \lambda(x) = \mathcal{A}(f) \).
4. Proof of Theorem.

The proof of the theorem will be carried out by the same way as for ordinary differential equations by using the lemmas in the above, cf. [1; pp. 91~95].

As noted in Lemma 2, the assumption in Lemma 4 is satisfied, and there exists an $R$-bounded solution $x(t)$ of (5) with the minimal norm $\lambda(x)$ by Lemma 5.

Now, we shall prove that for each $B(t, \phi) + g(t) \in H(A + f)$ the system

\begin{equation}
\dot{x}(t) = B(t, x_t) + g(t)
\end{equation}

has a unique $R$-bounded solution with the minimal norm. By Lemma 3 and the remark, the limit function of a suitable sequence $\{x(t + t_k)\}$ gives an $R$-bounded solution of (13) with the minimal norm $\alpha = \lambda(x)$.

Let both of $x^1(t)$ and $x^2(t)$ be $R$-bounded solutions of (13) with the minimal norm $\lambda(x^1) = \alpha = \lambda(x^2)$. Clearly $z(t) = \frac{1}{2} [x^1(t) - x^2(t)]$ is a solution of the homogeneous system (4) and $y(t) = \frac{1}{2} [x^1(t) + x^2(t)]$ is a solution of the system (13).

Since

\[ \frac{1}{2} \left[ |x^1(t)|^2 + |x^2(t)|^2 \right] = |y(t)|^2 + |z(t)|^2, \]

we have

\[ \frac{1}{2} \left[ \|x_t^1\|_*^2 + \|x_t^2\|_*^2 \right] = \|y_t\|_*^2 + \|z_t\|_*^2, \]

which implies

\[ \alpha^2 \leq \lambda(y)^2 \leq \alpha^2 - \inf_{t \in \mathbb{R}} \|z_t\|_*^2, \]

that is,

\[ \inf_{t \in \mathbb{R}} \|z_t\|_* = 0. \]

$z(t)$ is an $R$-bounded solution of (4) and satisfies a Lipschitz condition, and hence the relation (14) implies

\[ \inf_{t \in \mathbb{R}} \|z_t\| = 0. \]

Therefore, under the condition (3) we should have $z(t) \equiv 0$, which proves the uniqueness of an $R$-bounded solution with the minimal norm.

The fact that the unique $R$-bounded solution $x(t)$ of (5) with the minimal norm is almost periodic can be shown by the completely same way as in [1],...
and hence we shall omit the proof.

5. Autonomous system.

For an autonomous linear system

\begin{equation}
\dot{x}(t) = A(x_t)
\end{equation}

Hale [7] has shown that there exist two positively invariant spaces $S, U$ such that

\[ C([-h, 0], \mathbb{R}^n) = S \oplus U \]

with the properties:

(i) every solution of (15) starting from $S$ tends to zero as $t \to \infty$;

(ii) $\dim U < \infty$, and the solutions of (15) starting from $U$ are governed by an autonomous linear system of ordinary differential equations all of whose eigenvalues have non-negative real parts.

Let $x(t)$ be a solution of (15) defined on $\mathbb{R}$, and let $P_S, P_U = I - P_S$ denote the projections of $C([-h, 0], \mathbb{R}^n)$ into the spaces $S, U$, respectively. Then, by invariantness and linearity, we have

\begin{equation}
P_Sx_t = x(t; s, P_Sx_s) \quad \text{for all } t \geq s,
\end{equation}

and the same is true for $U$, where $x(t; s, \phi)$ denotes the solution of (15) through $(s, \phi)$.

If $x_t \in S$ for all $t \in \mathbb{R}$, then we can find two positive constants $M$ and $\alpha$ such that

\[ ||x_t|| \leq Me^{-\alpha(t-s)} ||x_s|| \quad \text{for all } t \geq s \]

(see [7]). Therefore,

\[ ||x_s|| \geq \frac{1}{M} e^{\alpha(t-s)} ||x_t|| \quad \text{for all } s \leq t, \]

which implies that $||x_s|| \to \infty$ as $s \to -\infty$ otherwise $x(t) \equiv 0$.

Thus, by noting the relation (16), a solution $x(t)$ of the system (15) defined on $\mathbb{R}$ is $\mathbb{R}$-bounded only if $P_Sx_t = 0$ for all $t \in \mathbb{R}$, that is, $x_t \in U$ for all $t \in \mathbb{R}$. Since on $U$ $x(t)$ is governed by a linear system of ordinary differential equations with constant coefficients, it is easy to see that every non-trivial $\mathbb{R}$-bounded solution of (15) satisfies the condition (3).

References

[2] L. Amerio, Soluzioni quasi-periodiche, o limitate, di sistemi differenziali non


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