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Remarks on a Universal Criterion for Liapunov Stability

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Introduction.

In a recent paper by the same authors, [4], a necessary and sufficient condition for the existence of a Liapunov function in the context of a preordered, bi-quasifiltered set was given.

A decisive rôle was played by the notion of an "admissible" quasifilter, *i. e.*, one admitting a nested, countable base. In [5], it had already been pointed out by the second author, that Liapunov stability does not necessarily imply the existence of a Liapunov function, the most typical case in question being that of "topological stability", a concept first introduced by J. Auslander in [1].

In a more recent paper, [2], J. Auslander has shown that in such cases, instead of using a single Liapunov function, one may use a family of "almost" Liapunov functions (*i. e.*, each satisfying slightly weaker conditions than a Liapunov function) to characterize stability.

The first purpose of this note is to prove that such "Liapunov families" (as they are called in [2]) may indeed serve to formulate a stability criterion which is universal in the general context indicated above, thus filling the gap that remained in the paper [4], concerning cases where suitable admissible quasifilters are not available.

In the second part, however, we show that, unless a single Liapunov function exists, the Liapunov family which takes its place is necessarily uncountable. This result further emphasizes the rôle of the admissible quasifilters in the context of Liapunov's direct method.

For all the terminology and notations used in this note the reader is referred to the preceding papers [4-6].

1. Given a set X endowed with a preorder Φ [reflexive and transitive relation] and a pair of quasifilters $\mathcal{D}$ and $\mathcal{E}$, we call a family $\mathcal{V}$ of functions $v\colon X\to\tilde{R}^+ := [0, \infty]$ a *Liapunov family* with respect to $(X, \Phi, \mathcal{D}, \mathcal{E})$ iff it satisfies the following conditions:

(1) $$\mathcal{E}\prec\mathcal{S}_{\mathcal{V}} := \bigcup\{\mathcal{S}_v | v\in\mathcal{V}\};$$

(2) $$\mathcal{S}_{\mathcal{V}}\prec\mathcal{D};$$

(3) $$\mathcal{S}_{\mathcal{V}} \prec \Phi \mathcal{S}_{\mathcal{V}}.$$

(In the case a single function v, this definition reduces, essentially, to that of a Liapunov function in the sense defined in [4], [5], with the only difference that (3) is actually weaker than the corresponding condition in [4], [5], which was the usual monotonicity property. Continuous functions with this property were called *para-Liapunov functions* by 0. Hájek in [3]; cf. also [6]).

Theorem 1. *(X, Φ) is $(\mathcal{D}, \mathcal{E})$-stable if and only if there exists a Liapunov family $\mathcal{V}$ for $(X, \Phi, \mathcal{D}, \mathcal{E})$.*

Note: In the more conventional setting of a (classical) dynamical system and stability of a set with respect to a neighborhood filter, this is essentially Auslander's theorem 5, [2].

The proof of sufficiency depends on the following proposition, the proof of which is rather obvious.

Proposition 1. *If X is a set, g a (possibly multiple-valued) function with domain X, and $\mathcal{A}$ and $\mathcal{B}$ are two quasifilters on X, then*

$$\mathcal{A} \prec \mathcal{B} \quad \textit{implies} \quad g\mathcal{A} \prec g\mathcal{B}.$$

*Here $g\mathcal{A}$ (for instance) denotes the quasifilter $\{gA \mid A \in \mathcal{A}\}$**.

Proof of sufficiency: First, recall [5] that:

$$\mathcal{S}_v := \{S_v^\beta \mid \beta > 0 \quad \text{such that} \quad S_v^\beta \neq \emptyset\},$$

where

$$S_v^\beta := \{x \in X \mid v(x) < \beta\}.$$

Then, applying successively the conditions (1), (3), and (2), together with proposition 1, we find:

$$\mathcal{E} \prec \mathcal{S}_{\mathcal{V}} \prec \Phi \mathcal{S}_{\mathcal{V}} \prec \Phi \mathcal{D};$$

thus, the relation $\prec$ being transitive, we have $\mathcal{E} \prec \Phi \mathcal{D}$, which is $(\mathcal{D}, \mathcal{E})$-stability.

Proof of necessity: Now suppose (X, Φ) is $(\mathcal{D}, \mathcal{E})$-stable. Then we define for each $D \in \mathcal{D}$ the function

$$v_D = \begin{Bmatrix} 0 \text{ on } \Phi D \\ 1 \text{ on } X \setminus \Phi D \end{Bmatrix}$$

and verify that the collection $\mathcal{V} = \{v_D \mid D \in \mathcal{D}\}$ satisfies the conditions in question.

Condition (1): Given $E \in \mathcal{E}$, choose $D \in \mathcal{D}$ such that $\Phi D \subset E$, and put $\beta = 1$, then $S_{v_D}^1 = \Phi D \subset E$, which proves (1).

Condition (2): Choose any v, say v_D, and any $\beta > 0$. If $\beta > 1$, $S_{v_D}^\beta = X \supset D$. If $\beta \leq 1$, $S_{v_D}^\beta = \Phi D \supset D$ (since Φ is reflexive). So in any case, $S_{v_D}^\beta$ contains D,

* $gA := \bigcup \{g(a) \mid a \in A\}$.

which proves (2).

From the definition of v_D and the transitivity of Φ, it follows that all the functions of $\mathcal{V}$ are nonincreasing under Φ. This obviously implies $S^{\beta}_{v_D} \supset \Phi S^{\beta}_{v_D}$, for any β and any $D \in \mathcal{D}$; hence (3).

Note: Auslander's original definition of a Liapunov family required the nonincreasing property for each individual function, rather than the weaker condition (3). We have in fact proved that the existence of a family satisfying the stronger condition is necessary for stability.

2. The following result shows, essentially, that unless a single Liapunov function exists, the Liapunov family is necessarily uncountable.

Theorem 2. *Let $\mathcal{V}$ be a Liapunov family with respect to $\Sigma = (X, \Phi, \mathcal{D}, \mathcal{E})$, and suppose $\mathcal{V}$ is countable and $\mathcal{D}$ is closed under finite intersections. Then there exists a Liapunov function with respect to Σ.*

Proof. Suppose $\mathcal{V} = (v_1, v_2, \cdots)$ is the family in question. The corresponding quasifilters $\mathcal{S}_{v_n}$ are admissible, *i. e.*, they have countable nested bases, say $\mathcal{B}_n = \{B_{n,k}\}_{k=1}^{\infty}$. Take, for instance, $B_{n,k} = S_{v_n}^{k^{-1} + \inf v_n}$. Let $i: N \to N^2$ be a bijection and denote $S_n = \bigcap \{B_{i(k)} | k = 1, \cdots, n\}$. Then $\mathcal{S} = \{S_n\}$ is an admissible quasifilter, satisfying the double relation

$$\mathcal{E} \prec \mathcal{S} \prec \mathcal{D}. \tag{4}$$

Indeed, the first part is a consequence of (1) and of the definition of $\mathcal{S}$, while the second part follows from (2) and the closedness of $\mathcal{D}$ under finite intersections.

Next we prove "$\mathcal{S}$-stability", *i. e.*,

$$\mathcal{S} \prec \Phi \mathcal{S}, \tag{5}$$

Given $S_n = \bigcap \{B_{i(k)} | k = 1, \cdots, n\}$, put $B_{i(k)} := B_{n_k, m_k}$; then there exists a β_k such that $B_{n_k, m_k} = S^{\beta_k}_{n_k}$, because $\mathcal{B}_{n_k}$ is a base, hence a subset of $\mathcal{S}_{n_k}$. Due to (3), $S^{\beta_k}_{n_k}$ contains some $\Phi S^{\alpha_k}_{p_k}$. Now select an element B_{p_k, l_k} of $\mathcal{B}_{p_k}$, contained in $S^{\alpha_k}_{p_k}$, thus implying $\Phi S^{\alpha_k}_{p_k} \supset \Phi B_{p_k, l_k}$. Then we have

$$\Phi B_{p_k, l_k} \subset \Phi B^{\alpha_k}_{p_k} \subset S^{\beta_k}_{n_k} = B_{n_k, m_k} = B_{i(k)}$$

Then, putting $m = \max\limits_{k=1, \cdots, n} i^{-1}(p_k, l_k)$, we finally obtain

$$\Phi S_m = \Phi \bigcap_{l=1}^{m} B_{i(l)} \subset \Phi \bigcap_{k=1}^{n} B_{p_k, l_k} = \bigcap_{k=1}^{n} \Phi B_{p_k, l_k} \subset \bigcap_{k=1}^{n} B_{i(k)} = S_n,$$

which proves (5).

The existence of a Liapunov function now follows from (4) and (5) by applying theorem C of [4].

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