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Structure and Characterizations of Certain Continuous Flows

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1. Introduction.

Ahmad [1] introduced and classified planar flows of characteristic $0^{\pm}$ (0^{-}) in terms of their critical points and he characterized planar flows of characteristic $0^{\pm}$ in terms of their critical points. The author [6], [7] characterized planar flows of characteristics $0^{\pm}$, 0^{-}, and 0 in terms of their critical points. In [2] Ahmad classified flows of characteristic $0^{\pm}$ (0^{-}) on locally compact phase spaces. The purpose of this paper is to characterize and study the structure of flows of characteristic 0 on Hausdorff phase spaces.

2. Basic Definitions.

A pair (X, π) consisting of a topological space X and a continuous mapping π from the product space $X\times R$ into X where R is the usual real space is called a *dynamical system* or (*continuous*) *flow* whenever the following conditions are satisfied.

1. $\pi(x, 0)=x$ for each $x\in X$.
2. $\pi(\pi(x, t), s)=\pi(x, t+s)$ for each $x\in X$ and $t, s\in X$.
3. π is continuous on $X\times R$.

We denote $\pi(x, t)$ by xt for brevity and call X the *phase space*. Throughout this paper, the phase spaces will be Hausdorff spaces.

We denote the (*positive, negative*) *trajectory* through $x\in X$ by $(C^{+}(x), C^{-}(x))$ $C(x)$ and $K^{+}(x)=\overline{C^{+}(x)}$, $K^{-}(x)=\overline{C^{-}(x)}$, and $K(x)=\overline{C(x)}$. The (*positive, negative*) *limit set* is denoted by $(L^{+}(x), L^{-}(x))$ $L(x)$. The (*positive, negative*) *prolongation* and (*positive, negative*) *prolongational limit set* are denoted by $(D^{+}(x), D^{-}(x))$ $D(x)$ and $(J^{+}(x), J^{-}(x))$ $J(x)$, respectively.

We let $\mathcal{N}(x)$ and $\mathcal{N}(M)$ denote the neighborhood filters of $x\in X$ and $M\subset X$.

A set $M\subset X$ is called *positively stable* if for every $U\in\mathcal{N}(M)$ there exists a $V\in\mathcal{N}(M)$ such that $V=C^{+}(V)\subset U$, A point x in X is *positively* (*weakly*) *attracted* to $M\subset X$ if the net (xt) for $t\geq 0$ is (frequently) ultimately in each neighborhood of M. If the set $A^{+}(M)$ $(A_{w}^{+}(M))=\{x$: x is positively (weakly) attracted to $M\}$ is a neighborhood of M, then M is called a *positive* (*weak*) *attractor*. We call M *positively asymptotically stable* when M is both positively stable and a positive attractor. The corresponding negative and bilateral versions of the terms above are defined similarly. For brevity, we shall drop the adjective "positive" from the four terms defined above.

A flow (X,π) is called *(negatively, bilaterally) dispersive* if $(J^-(x)=\phi$, $J(x)=\phi)$ $J^+(x)=\phi$ for each $x\in X$. A point x in X is called *nonwandering* if $x\in J^+(x)$. A point x in X is said to be *(positively, negatively) Lagrange stable* if and only if $(K^+(x), K^-(x))$ $K(x)$ is compact. We call a point x in X *(positively, negatively) Poisson stable* whenever $(x\in L^+(x)$, $x\in L^-(x))$ $x\in L^+(x)\cap L^-(x)$. A point x in X is called *recurrent* if and only if for any $V\in\mathfrak{N}(x)$ there is a $t\geq 0$ such that $V\cap y[0,t]\neq\phi$ for every $y\in C(x)$. A set $M\subset X$ is said to be *(positively, negatively) minimal* if and only if it is closed and is (positively, negatively) invariant and contains no nonempty proper subset with these respective properties.

For the basic properties of dynamical systems used in this paper we refer to [4] and [5].

3. Flows of Characteristic 0.

A flow (X,π) is said to be of characteristic 0^+ (0^-) if $D^+(x)=K^+(x)$ $(D^-(x)=K^-(x))$ for each x in X, or equivalently, if $J^+(x)=L^+(x)$ $(J^-(x)=L^-(x))$ for each x in X. A flow having characteristics 0^+ and 0^- is called a flow of characteristic $0^{\pm}$. We say (X,π) has characteristic 0 whenever $D(x)=K(x)$ for each x in X.

Proposition 1. *Let (X,π) have characteristic 0. Then (X',π') has characteristic 0 where X' is an invariant subset of X and $\pi'=\pi|X'$.*

Proof. Let $x\in X'$. Then $D'(x)=J'(x)\cup C(x)\subset(J(x)\cap X')\cup C(x)\subset(K(x)\cap X')\cup C(x)=L'(x)\cup C(x)=K'(x)$.

Proposition 2. *The following are equivalent for a flow (X,π).*

(a) *(X,π) has characteristic 0.*

(b) *For each $x\in X$, $J^+(x)=J^-(x)=\begin{cases}K(x)\\ \phi\end{cases}$.*

(c) *For each $x\in X$, $J^+(x)=J^-(x)=\begin{cases}L(x)\\ C(x).\end{cases}$*

Proof. We first show that (a) implies (b). Let $J^+(x)\neq\phi$ for some $x\in X$. If $y\in J^+(x)$, then $x\in J^-(y)\subset D(y)=K(y)\subset J^+(x)$. Hence, $K(x)\subset J^+(x)$. Since $J^+(x)\subset D(x)=K(x)$, we have $J^+(x)=K(x)$. Furthermore, $x\in J^+(x)$ implies $x\in J^-(x)$, and hence, $K(x)\subset J^-(x)\subset D(x)=K(x)$. Similary, $J^-(x)\neq\phi$ for some $x\in X$ implies $J^+(x)=J^-(x)=K(x)$. Finally, whenever $J(x)=\phi, J^+(x)=J^-(x)=\phi$.

Next, we show that (b) implies (c). If $J^+(x)=J^-(x)=\phi$ for some $x\in X$, then $J^+(x)=J^-(x)=\phi=L(x)$. If $J(x)\neq\phi$ and $K(x)=C(x)$ for some $x\in X$, then

$J^+(x)=J^-(x)=C(x)$. If $J(x)\neq\phi$ and $K(x)\neq C(x)$ for some $x\in X$, then $L(x)\neq\phi$. Furthermore, for $y\in L(x)$ we have $x\in J(y)\subset D(y)=C(y)\cup K(y)=K(y)\subset L(x)$ since $J(y)=K(y)$. Thus, $J^+(x)=J^-(x)=K(x)\subset L(x)$, and hence, $J^+(x)=J^-(x)=L(x)$.

Finally, we show that (a) follows from (c). For any $x\in X$, $D(x)=C(x)\cup J(x)\subset C(x)\cup L(x)=K(x)$. Hence, $D(x)=K(x)$.

Proposition 3. *Let (X,π) have characteristic 0. If $L^+(x)\neq\phi$ $(L^-(x)\neq\phi)$ for some $x\in X$, then $J^+(x)=J^-(x)=L^+(x)$ $(J^+(x)=J^-(x)=L^-(x))$.*

Proof. Let $y\in L^+(x)$ for some x in X. Then $y\in J^+(x)$ and $x\in J^-(y)\subset D(y)=K(y)\subset L^+(x)$. Thus, $J^+(x)\subset D(x)=K(x)\subset L^+(x)$, and so, $J^+(x)=L^+(x)$. By Proposition 2, $J^-(x)=L^+(x)$.

Corollary 3.1. *Let (X,π) have characteristic 0. If $J^+(x)\neq\phi$ for some $x\in X$, then*

$$J(x)=J^+(x)=J^-(x)=D(x)=D^+(x)=D^-(x)=K(x)$$
$$=\begin{cases}K^+(x)=K(x)=L^+(x) \text{ if } L^+(x)\neq\phi.\\ K^-(x)=K(x)=L^-(x) \text{ if } L^-(x)\neq\phi.\end{cases}$$

Proposition 4. *If (X,π) has characteristic 0, then $L^+(x)$ is minimal for each $x\in X$.*

Proof. $L^+(x)$ is trivially minimal whenever it is empty. Let $y\in L^+(x)$ for some $x\in X$. Then $L^+(x)\subset J^+(x)\subset J^+(y)\subset D(y)=K(y)$, and hence, $K(y)=L^+(x)$. Thus, $L^+(x)$ is minimal (see 4.15 of [4]).

Corollary 4.1. *Let (X,π) be of characteristic 0 and X be locally compact. If $L^+(x)$ $(L^-(x))$ is compact for some $x\in X$, then $L^+(x)$ $(L^-(x))$ is positively and negatively minimal and each point of $L^+(x)$ $(L^-(x))$ is recurrent.*

Proof. See Theorem 4.8 of [5] and Theorem 4.22 of [4].

Proposition 5. *For a flow (X,π) of characteristic 0 with locally compact phase space the following are equivalent for $x\in X$.*

(a) *$L^+(x)$ is compact minimal.*

(b) *$L^+(y)\neq\phi$ for each $y\in L^+(x)$.*

(c) *$L^+(y)=L^+(x)$ for each $y\in L^+(x)$.*

The negative limit set version also holds.

Proof. The equivalence of the statements follows trivially whenever $L^+(x)=\phi$ for some $x\in X$. Let $L^+(x)\neq\phi$ for some $x\in X$. If $L^+(x)$ is compact, then $L^+(y)\neq\phi$ for each $y\in L^+(x)$ (see 3.6.1 of [4]). Thus, (a) implies (b). If (b) holds, then for $y\in L^+(x)\subset J^+(x)$ we have $x\in J^-(y)$. By Proposition 3, $x\in L^+(y)$, and hence, $L^+(y)=L^+(x)$. Finally, if (c) holds, then $x\in L^+(x)=L^+(y)$ for each $y\in L^+(x)$. Hence, $L^+(x)\subset A_w^+(x)$ and $L^+(x)$ is compact minimal (see 2.4.2 and 4.6 of [5]).

Corollary 5.1. *If* (X, π) *is of characteristic* 0 *with locally compact phase space and* $L^+(x)$ *is a nonempty compact minimal set for some* $x \in X$, *then* $L(x)=L^+(x)=L^-(x)=K(x)=K^+(x)=K^-(x)=J(x)=J^+(x)=J^-(x)=D(x)=D^+(x)=D^-(x)$.

Corollary 5.2. *Let* (X, π) *be of characteristic* 0 *with locally compact phase space and let* $L^+(x)$ *be a nonempty compact minimal set for each* $x \in X$. *Then* (X, π) *is of characteristics* $0^+, 0^-$, *and* $0^\pm$.

We give the following notation for convenience. For a flow (X, π) we let

$$M_1 = \{x: L^+(x) \neq \phi \text{ and } L^-(x) = \phi\},$$
$$M_2 = \{x: L^+(x) = \phi \text{ and } L^-(x) \neq \phi\},$$
$$M_3 = \{x: L(x) = \phi \text{ and } J^+(x) \neq \phi\}, \text{ and}$$
$$M_4 = \{x: L^+(x) = L^-(x) = J^+(x)\}.$$

Theorem 6. *Let* (X, π) *have characteristic* 0. *The sets* M_1, M_2, M_3, *and* M_4 *are pairwise disjoint sets whose union is* X. *The restriction of the flow to*

(i) $M_1 \cup M_4$ *is of characteristics* 0 *and* 0^+, *and is of characteristics* 0^- *and* $0^\pm$ *if and only if* $M_1 = \phi$,

(ii) $M_2 \cup M_4$ *is of characteristics* 0 *and* 0^-, *and is of characteristics* 0^+ *and* $0^\pm$ *if and only if* $M_2 = \phi$,

(iii) M_4 *is of characteristics* 0 *and* $0^\pm$, *and*

(iv) M_3 *is only of characteristic* 0 *provided it is not dispersive.*

Proof. That M_1, M_2, M_3 and M_4 are pairwise disjoint sets whose union is X follows from Proposition 3 and its corollary. According to Proposition 1, each restriction has characteristic 0.

Let $\pi' = \pi | M_1 \cup M_4$. Then $J'^+(x) = L'^+(x)$ for each $x \in M_1 \cup M_4$ since $J'^+(x) \subset J^+(x) \cap (M_1 \cup M_4) = L^+(x) \cap (M_1 \cup M_4) = L'^+(x)$. Hence, the restricted flow has characteristic 0^+. Furthermore, for $x \in M_4$, $J'^-(x) = J'^+(x) = L'^-(x)$ and for $x \in M_1$, $J'^-(x) \neq \phi$ while $L'^-(x) = \phi$. Consequently, the restricted flow does not have characteristic 0^- or $0^\pm$ if and only if $M_1 \neq \phi$.

The proof of (ii) follows similarly and (iii) is a result of (i) and (ii).

Finally, if the flow (M_3, π') where $\pi' = \pi | M_3$ is not dispersive, then it is not negatively dispersive. In this case, since $L^+(x) = L^-(x) = \phi$ for each $x \in M_3$, (M_3, π') is not of characteristics $0^+, 0^-$, and $0^\pm$.

Corollary 6.1. *Let* (X, π) *be of characteristic* 0. *Then* (X, π) *has characteristic* 0^+ (0^-) *if and only if* $M_2 = M_3 = \phi$ $(M_1 = M_3 = \phi)$. *Furthermore,* (X, π) *has characteristic* $0^\pm$ *if and only if* $X = M_4$.

Corollary 6.2. *A flow* (X, π) *of characteristic* 0 *with locally compact phase space is of characteristic* $0^\pm$ *if and only if each nonwandering point is*

Lagrange stable.

Proof. Let (X,π) have characteristic $0^{\pm}$ and let $x\in J^{+}(x)$ for some $x\in X$. Then $L^{+}(x)=J^{+}(x)\neq\phi$. For $y\in L^{+}(x)$ we have $J^{+}(x)\subset J^{+}(y)=L^{+}(y)$ so that $L^{+}(y)\neq\phi$. By Proposition 5 and Corollary 5.1, $K(x)$ is compact.

Conversely, if $J^{+}(x)=\phi$, then $J^{+}(x)=J^{-}(x)=L^{+}(x)=L^{-}(x)$. Let $J^{+}(x)\neq\phi$. Then $J^{+}(x)=K(x)$ which is compact. Hence, $L^{+}(x)\neq\phi$ and $L^{-}(x)\neq\phi$ yielding $X=M_4$.

Corollary 6.3. *Let (X,π) have a compact phase space. Then the characteristic $0^{+}, 0^{-}, 0^{\pm}$, and 0 properties are equivalent. Furthermore, in this case $X=M_4$ and $K(x)$ is a compact minimal bilaterally stable set for each $x\in X$.*

Proof. The equivalence of the characteristic $0^{+}, 0^{-}$, and $0^{\pm}$ properties was shown by Ahmad in [2]. Any flow of characteristic $0^{\pm}$ is of characteristic 0. Let (X,π) be of characteristic 0. Then for each $x\in X$, $L^{+}(x)\neq\phi$ and $L^{-}(x)\neq\phi$ since X is compact. Hence, $X=M_4$, and so, (X,π) is of characteristic $0^{\pm}$. The remainder of the proof follows from Proposition 5, Corollary 5.1, and Theorem 4.7 of [2].

Corollary 6.4. *Let (X,π) be of characteristic 0 where X is metric and either locally compact or complete. Then $X=\overline{M_4}$.*

Proof. Let X' be the closed set $\{x: J^{+}(x)\neq\phi\}$ (see 4.2.3 of [4]). Now $L(x)\subset X'$ for each $x\in X$ since $y\in L(x)$ implies that $x\in J(y)$. Thus, $L'(x)=L(x)\cap X'=L(x)$ for each $x\in X'$. Also for any $x\in X'$, $J(x)=K(x)=K'(x)=J'(x)$. Each point of X' is nonwandering so that the set $M_4\cap X'$ of bilaterally Poisson stable points is dense in X' (see 4.6 of [4]). Hence, $X=\overline{M_4}$.

Example. There are flows for which the sets M_1, M_2, M_3, *and* M_4 are all nonempty. Let X be the union of the torus Y and the plane R^2. Define $\pi: X\times R\to X$ as follows. On R^2 define π by the system of differential equations

$$\dot{r}=-r^2\sin\theta$$
$$\dot{\theta}=1,$$

$r\geq 0$ (see Example 2 of [6]) and on Y define π by the planar system

$$\dot{x}=f(x, y)$$
$$\dot{y}=\alpha f(x, y),$$

α irrational, where $f(x, y)=f(x+1, y+1)=f(x, y+1)=f(x+1, y)$, $f(x, y)>0$ if x and y are not both zero (mod 1), and $f(0,0)=0$ (see p. 56 of [4] and p. 33 of [3]). Let p be the critical point on the torus. The flow restricted to the locally compact phase space $X-\{p\}$ is of characteristic 0 and the sets M_1, M_2, M_3, and M_4 are nonempty.

Theorem 7. *Let (X,π) be a flow with locally compact phase space. Then (X,π) is of characteristic 0 if and only if*

(i) *each compact minimal set is bilaterally stable and*

(ii) $J(x)\subset K(x)$ *for each* x *not in a compact minimal set.*

Proof. *Let* (X,π) be of characteristic 0 and let H be compact minimal. Then $D(H)=\cup\{D(x): x\in H\}=\cup\{K(x): x\in H\}=H$ since $K(x)=H$ for each $x\in H$ (see 4.15 of [4]). Hence, H is bilaterally stable by Ura's Theorem (see 2.1 of [6]). Condition (ii) follows by Proposition 2.

Conversely, if $x\in H$ where H is compact minimal, then by Ura's Theorem we have $D(x)\subset D(K(x))=K(x)$ since $H=K(x)$, and hence, $D(x)=K(x)$. On the other hand, if x is not in a compact minimal set, then $D(x)\subset K(x)$. In either case $D(x)=K(x)$. Therefore, (X,π) is of characteristic 0.

Corollary 7.1. *Let* (X,π) *be of characteristic* 0 *with locally compact phase space. Each compact minimal set has a neighborhood of Poisson stable points.*

Proof. Any compact minimal set H has a compact neighborhood N since X is locally compact. The bilateral stability of H implies that there is an invariant neighborhood V of H contained in N. For any point x in V, $L^+(x)\neq\phi$ and $L^-(x)\neq\phi$, and hence, each point of V is Poisson stable.

Theorem 8. *Let* (X,π) *be a flow of characteristic* 0 *with locally compact phase space. A closed connected invariant set* M *with compact boundary is either a component of* X *or is not isolated from nonempty compact minimal sets.*

Proof. Suppose M is a closed connected invariant set which is not a component of X. Since the boundary of M is a compact invariant set, $L^+(x)\neq\phi$ and $L^+(x)\subset\partial M$ for each $x\in\partial M$. Thus, ∂M is the union of compact minimal bilaterally stable sets by virtue of Proposition 5 and Theorem 7, and hence, M is bilaterally stable. Since X is locally compact and ∂M is compact, there is a compact $V\in\mathfrak{N}(\partial M)$. Let $U\in\mathfrak{N}(M)$. The bilateral stability of M implies that there is an invariant set $W\in\mathfrak{N}(M)$ such that $C(W)\subset U\cap(V\cup M)$. For every $x\in W-M$, $L^+(x)$ is nonempty compact minimal. Thus, M is not isolated from nonempty compact minimal sets.

Proposition 9. *Let* (X,π) *be a flow of characteristic* 0. *Then for each* $x\in X$,

$$A_w^+(L^+(x))=L^+(x),\quad A_w^+(L^-(x))=L^-(x),\quad \textit{and}\quad A_w^+(C(x))=K(x).$$

Proof. If $L^+(x)=\phi$ for some $x\in X$, then the first equality follows trivially. Let $L^+(x)\neq\phi$ for some $x\in X$. Since we already have $L^+(x)\subset A_w^+(L^+(x))$, let $y\in A_w^+(L^+(x))$. Either the net (yt) is frequently in $L^+(x)$ or $L^+(y)\cap L^+(x)\neq\phi$. If (yt) is frequently in $L^+(x)$, then $y\in L^+(x)$. If some point z is in $L^+(y)\cap L^+(x)$, then we have $y\in J^+(z)=L^+(z)\subset L^+(x)$. Thus, $A_w^+(L^+(x))\subset L^+(x)$. We can obtain $A_w^+(L^-(x))=L^-(x)$ similarly.

Finally, if $L^+(x)\neq\phi$ for some $x\in X$, then $A_w^+(C(x))\subset A_w^+(L^+(x))=L^+(x)=K(x)\subset A_w^+(C(x))$. Similarly, $L^-(x)\neq\phi$ yields $A_w^+(C(x))=K(x)$. Let $L(x)=\phi$. Either $J^+(x)=J^-(x)=C(x)$ or $J^+(x)=J^-(x)=\phi$. First, suppose that $J^+(x)=J^-(x)=C(x)$. Then for each $y\in A_w^+(C(x))$ we have $J^+(y)\subset C(x)$ since $J^+(A_w^+(C(x)))\subset J^+(C(x))=J^+(x)=C(x)$ (see 2.19 of [5]). If $J^+(y)\neq\phi$, then $y\in J^+(y)\subset C(x)$. If $J^+(y)=\phi$, then $L^+(y)=\phi$ and (yt) is frequently in $C(x)$. Thus, $A_w^+(C(x))\subset C(x)$, and hence, $A_w^+(C(x))=C(x)=K(x)$. Next, suppose that $J^+(x)=J^-(x)=\phi$. Then $L^+(y)=J^+(y)=\phi$ for each $y\in A_w^+(C(x))$ since $J^+(A_w^+(C(x)))\subset J^+(C(x))=J^+(x)=\phi$. Thus (yt) is frequently in $C(x)$ for each $y\in A_w^+(C(x))$ implying that $A_w^+(C(x))=C(x)=K(x)$.

Corollary 9.1. *Let (X,π) be of characteristic 0. Then $A^+(L^+(x))=L^+(x)$, $A^+(L^-(x))=L^-(x)$, and $A^+(C(x))=K(x)$ for each $x\in X$.*

Proof. For any invariant subset M of X, $M\subset A^+(M)\subset A_w^+(M)$. Hence, each statement holds trivially.

Theorem 10. *A necessary and sufficient condition for a flow (X,π) to be of characteristic 0 is that $A^+(C(x))=D(x)$ for each $x\in X$.*

Proof. The necessity of the condition follows from Proposition 9. Conversely, if $J(x)=\phi$ for some $x\in X$, then $D(x)=C(x)=K(x)$. Let $J(x)\neq\phi$ for some $x\in X$. If $y\in J(x)$, then $x\in J(y)\subset A^+(C(y))$ which implies that $J(x)\subset J(A^+(C(y)))\subset J(C(y))=J(y)$ (the bilateral version of 2.19, [5] follows easily). Also, $y\in J(x)$ implies that $y\in A^+(C(x))$, and so, $J(y)\subset J(A^+(C(x)))\subset J(C(x))=J(x)$. Thus, $J(x)=J(y)$ for any $y\in J(x)$ which implies $x\in J(y)=J(x)$. Consequently, $A^+(C(x))=J(x)$ whenever $J(x)\neq\phi$. Next, let $z\in J(x)$. Then either $L^+(z)\cap C(x)\neq\phi$ or $C(z)=C(x)$. In either case, $K(x)\subset K(z)$. On the other hand, $z\in J(x)$ implies that $K(z)\subset K(x)$, and thus, that $K(x)=K(z)$ for each $z\in J(x)$. Finally, $D(x)=A^+(C(x))=J(x)=\cup\{K(z):\ z\in J(x)\}=K(x)$. We now have $D(x)=K(x)$ for each $x\in X$.

Corollary 10.1. *A flow (X,π) is of characteristic 0 if and only if $A^+(M)=D(M)$ for each invariant set $M\subset X$.*

Corollary 10.2. *Let (X,π) be of characteristic 0 (with X regular). Then a compact (closed) invariant set is asymptotically stable if and only if it is open.*

Proof. An open invariant set is obviously asymptotically stable. If M is a compact (closed) invariant asymptotically stable set, then we can show that $A^+(M)=M$. For let $y\in A^+(M)$. Then $L^+(y)\cap M\neq\phi$ or $C(y)\subset M$. If $z\in L^+(y)\cap M$, then $y\in J^+(y)\subset J^+(z)\subset D^+(M)=M$ (see 1.9 and 1.15 of [5]). In either case, $y\in M$, and hence, $A^+(M)\subset M$ or $A^+(M)=M$.

Corollary 10.3. *Let (X, π) be of characteristic* 0 *(with X regular). Then a compact (closed) connected invariant set is asymptotically stable if and only if it is a component of X. Furthermore, if X is connected, there are no compact (closed) connected invariant asymptotically stable proper subsets of X.*

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