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Solutions Sets of Non-linear Integral Equations

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N. Kikuchi and S. Nakagiri [2] proved an existence theorem of solutions of a non-linear integral equation of Volterra-type

$$x(t)=f(t)+\int_{0}^{t}a(t,s)g(s,x(s))ds. \tag{1}$$

Under the same assumptions as in [2], in this note we investigate some topological properties of the set of solutions of the equation (1).

1. Assume that $N=\{1,2,\cdots\}$, R is the set of real numbers, J is a compact interval in R, $|\cdot|$ is the Euclidean norm in R^{d}, and φ is a convex N-function which satisfies the condition Δ_{2} (cf. [3]). Let $L_{\varphi}(J)=L_{\varphi}(J,R^{d})$ be the Orlicz space of all L-measurable functions $u: J\longrightarrow R^{d}$ for which the number

$$\|u\|_{\varphi}=\inf\left\{r>0:\int_{J}\varphi\left(\left|\frac{u(s)}{r}\right|\right)ds\leq 1\right\}<\infty.$$

The adjoint space of $L_{\varphi}(J)$ we denote by $L_{\varphi}^{*}(J)$. Obviously $L_{\varphi}^{*}(J)=L_{\psi}(J)$, where ψ is the N-function defined by the formula $\psi(v)=\sup\{uv-\varphi(u): u\geq 0\}$ (cf. [3]). We shall introduce the product space

$$L_{\varphi}^{*}(J)^{d}=L_{\varphi}^{*}(J)\oplus\cdots\oplus L_{\varphi}^{*}(J).$$

Furthermore, let W be an open convex set in R^{d} and let I be an open interval in R containing 0. Analogously as in [2], we introduce the following assumptions:

(i) $f: I\longrightarrow W$ is a continuous function.

(ii) $g: I\times W\longrightarrow R^{d}$ is a function such that

1° for each fixed $x\in W$, the function $t\longrightarrow g(t,x)$ is L-measurable on I;

2° for each fixed $t\in I$, the function $x\longrightarrow g(t,x)$ is continuous on W;

3° for each compact set $K\subset W$ and each compact interval $J\subset I$ there exists a measurable real-valued function $m\in L_{\varphi}(J,R)$ such that $|g(t,x)|\leq m(t)$ for $(t,x)\in J\times K$.

(iii) $(t,s)\longrightarrow a(t,s)$ is a mapping of $I\times I$ into the space M^{d} of linear operators on R^{d} such that

1° for each compact interval $J\subset I$ and each t in I the mapping $A: L_{\varphi}(J)\longrightarrow R^{d}$ defined by $A: x(\cdot)\longrightarrow\int_{J}a(t,s)x(s)ds$ is a bounded linear mapping;

2° the mapping $I \longrightarrow L_\varphi^*(J)^d$ defined by $t \longrightarrow a(t, \cdot)$ is continuous in the weak*-topology on $L_\varphi^*(J)^d$.

Let $J \subset I$ be a compact interval, and let $L_1(J)$ be the space of all L-integrable functions $u: J \longrightarrow R^d$ with the norm

$$\|u\|_1 = \int_J |u(s)| ds.$$

Put

$$B_\varphi^m(J) = \{x \in L_\varphi(J) : |x(t)| \leq m(t) \text{ for almost every } t \in J\}$$

and

$$H(x)(t) = f(t) + \int_0^t a(t, s) x(s) ds \quad \text{for each } x \in B_\varphi^m(J) \text{ and } t \in J.$$

In the same way as in [2], we prove that

(iv) for each fixed $x \in B_\varphi^m(J)$ the function $t \longrightarrow H(x)(t)$ is continuous on J.

(v) H is a mapping of $B_\varphi^m(J)$ into $L_1(J)$, and the set $H(B_\varphi^m(J))$ is compact in the strong topology on $L_1(J)$.

(vi) for any $t \in J$ the mapping $a(t, \cdot) \in L_\varphi^*(J)^d$, $\sup\{\|a(t, \cdot)\|_\phi : t \in J\} < \infty$.

Moreover, we introduce the following definition (cf. [1]):

A subset Q of a metric space X is called a compact R_δ iff Q is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

2. Choose a positive number c such that $[0, c] \subset I$. We can find a positive number h and some convex compact subset K of W such that every $x(t)$, $|x(t) - f(t)| \leq h$ for $0 \leq t \leq c$, satisfies $x(t) \in K$ for $0 \leq t \leq c$. Moreover, choose a number p, $0 < p \leq c$, such that $2\sup\{\|a(t, \cdot)\|_\phi : t \in [0, c]\} \cdot \|m\chi_{[0,p]}\| \leq h$, where m is a measurable function defined in (ii) corresponding to the pair $K, [0, c]$.

Theorem. *Let f, g and a satisfy, respectively, assumptions* (i), (ii) *and* (iii), *and let $J = [0, p]$. Then the set V of all solutions of the equation* (1) *defined on J is a compact R_δ in $L_1(J)$.*

Proof. By the Dugundji extension theorem there is a continuous function $r: R^d \longrightarrow K$ such that $r(x) = x$ for every $x \in K$. Put

$$G(x)(t) = f(t) + \int_0^t a(t, s) g(s, r(x(s))) ds \quad \text{for } t \in J \text{ and } x \in L_1(J).$$

From the assumptions (i)-(iii) it follows that for any fixed $x \in L_1(J)$ the function $t \longrightarrow G(x)(t)$ is continuous on J, and therefore $G(x) \in L_1(J)$. Moreover,

$$|G(x)(t) - f(t)| = \left| \int_0^t a(t, s) g(s, r(x(s))) ds \right| \leq 2\|a(t, \cdot)\|_\phi \|m\chi_J\|_\varphi \leq h,$$

i.e. $G(x)(t) \in K$ for each $t \in J$, $x \in L_1(J)$.

Assume now that a sequence (x_n), $x_n \in L_1(J)$, converges in $L_1(J)$ to $x_0 \in L_1(J)$. Suppose that $\|G(x_n)-G(x_0)\|_1$ is not convergent to 0 as $n \to \infty$. Then there are $\varepsilon > 0$ and a subsequence (x_{n_k}) such that

$$(2) \qquad \|G(x_{n_k})-G(x_0)\|_1 > \varepsilon \quad \text{for} \quad k \in N.$$

Since $\lim_{k\to\infty} \|x_{n_k}-x_0\|_1=0$, we can find a subsequence $(x_{n_{k_j}})$ such that $\lim_{j\to\infty} x_{n_{k_j}}(s)=x_0(s)$ for almost every t in J. Let $y_j = x_{n_{k_j}}$. By (ii), for any $t \in J$ we have

$$\lim_{j\to\infty} a(t,s)g(s,r(y_j(s)))=a(t,s)g(s,r(x_0(s)))$$

for almost every s in J. Furthermore, $|a(t,s)g(s,r(y_j(s)))| \leq |a(t,s)|m(s)$, where $|a(t,\cdot)|m(\cdot) \in L_1(J,R)$, and hence by the Lebesgue theorem we obtain

$$\lim_{j\to\infty} \int_0^t a(t,s)g(s,r(y_j(s)))ds=\int_0^t a(t,s)g(s,r(x_0(s)))ds,$$

i.e. $\lim_{j\to\infty} G(y_j)(t)=G(x_0)(t)$ for $t \in J$. Since $|G(y_j)(t)| \leq |f(t)|+h$ for $j \in N$ and $t \in J$, the Lebesgue theorem proves that $\lim_{j\to\infty} \|G(y_j)-G(x_0)\|_1=0$, in contradiction with (2). Consequently, the mapping $G: L_1(J) \longrightarrow L_1(J)$ is continuous.

For any $n \in N$ and $x \in L_1(J)$ let us put $p_n = p/n$ and

$$G_n(x)(t)=\begin{cases} f(0) & \text{for} \quad 0 \leq t \leq p_n \\ G(x)(t-p_n) & \text{for} \quad p_n \leq t \leq p. \end{cases}$$

Obviously, G_n is a continuous mapping of $L_1(J)$ into $L_1(J)$. We shall show that

$$(3) \qquad \lim_{n\to\infty} \|G_n(x)-G(x)\|_1=0 \text{ uniformly for } x \in L_1(J).$$

Suppose that (3) does not hold. Then there exist $\varepsilon > 0$ and sequences (n_k), (x_k), $x_k \in L_1(J)$, such that

$$(4) \qquad \|G_{n_k}(x_k)-G(x_k)\|_1 > \varepsilon \qquad \text{for} \quad k \in N.$$

Since $G(L_1(J)) \subset H(B_\varphi^m(J))$ and $H(B_\varphi^m(J))$ is compact in $L_1(J)$, we can find a subsequence $(G(x_{k_j}))$ which converges in $L_1(J)$ to a continuous function $u \in H(B_\varphi^m(J))$. Put $\widetilde{G}_j = G_{n_{k_j}}$, $q_j = p_{n_{k_j}}$ and $y_j = x_{k_j}$. Then

$$\begin{aligned}
&\|\widetilde{G}_j(y_j)-G(y_j)\|_1 \leq \|(\widetilde{G}_j(y_j)-G(y_j))\chi_{[0,q_j]}\|_1+\|(\widetilde{G}_j(y_j)-G(y_j))\chi_{[q_j,p]}\|_1 \\
&=\|(f(0)-G(y_j))\chi_{[0,q_j]}\|_1+\|(\widetilde{G}_j(y_j)-G(y_j))\chi_{[q_j,p]}\|_1 \leq \|(f(0)-u)\chi_{[0,q_j]}\|_1 \\
&\quad +\|(G(y_j)-u)\chi_{[0,q_j]}\|_1+\|(G(y_j)(\cdot-q_j)-u(\cdot-q_j))\chi_{[q_j,p]}\|_1 \\
&\quad +\|(u(\cdot-q_j)-u)\chi_{[q_j,p]}\|_1+\|(u-G(y_j))\chi_{[q_j,p]}\|_1 \leq \|(f(0)-u)\chi_{[0,q_j]}\|_1 \\
&\quad +\|(u(\cdot-q_j)-u)\chi_{[q_j,p]}\|_1+3\|G(y_j)-u\|_1,
\end{aligned}$$

which implies

$$\lim_{j\to\infty} \|G_{n_{k_j}}(x_{n_{k_j}})-G(x_{n_{k_j}})\|_1=0,$$

in contradiction with (4). This proves (3).

Put $T_n=I-G_n$ for $n\in N$, where I denotes the identity mapping of $L_1(J)$ into $L_1(J)$. Obviously, T_n is a continuous mapping of $L_1(J)$ into $L_1(J)$.

Assume that $y\in L_1(J)$. We define a finite sequence (x_k), $k=1,\cdots,n$, of continuous functions by the formulas

$$x_1(t)=y(t)+f(0) \quad \text{for} \quad 0\le t\le p_n$$

$$x_{k+1}(t)=\begin{cases} x_k(t) & \text{for} \quad 0\le t\le kp_n \\ y(t)+f(t-p_n)+\int_0^{t-p_n} a(t-p_n, s)g(s, r(x_k(s)))ds & \end{cases}$$

$$\text{for} \quad kp_n\le t\le (k+1)p_n, \quad k=1,\cdots,n-1.$$

We see that

$$x_k(t)=y(t)+f(0) \quad \text{for} \quad 0\le t\le p_n$$

$$x_k(t)=y(t)+f(t-p_n)+\int_0^{t-p_n} a(t-p_n, s)g(s, r(x_k(s)))ds$$

$$\text{for} \quad p_n\le t\le kp_n$$

$$x_{k+1}|_{[0,kp_n]}=x_k \quad \text{and} \quad x_k\in L_1([0,kp_n]).$$

Consequently, $x_n\in L_1(J)$ and $T_n(x_n)=y$. Conversely, if $T_n(x)=y$ and $x\in L_1(J)$, then $x|_{[0,kp_n]}=x_k$ for $k=1,\cdots,n$, and therefore $x=x_n$. This proves that $T_n\colon L_1(J)\longrightarrow L_1(J)$ is a bijection.

Now we assume that $\lim_{j\to\infty} \|T_n(x_j)-T_n(x_0)\|_1=0$, where $x_j, x_0\in L_1(J)$. Since $x_j(t)=T_n(x_j)(t)+f(0)$ for $0\le t\le p_n$, $x_j\chi_{[0,p_n]}\longrightarrow x_0\chi_{[0,p_n]}$ in $L_1(J)$ when $j\to\infty$. Further,

$$x_j(t)=T_n(x_j)(t)+G(x_j)(t-p_n)=T_n(x_j)(t)+G(x_j\chi_{[0,p_n]})(t-p_n)$$

for $p_n\le t\le 2p_n$, and $G(x_j\chi_{[0,p_n]})\longrightarrow G(x_0\chi_{[0,p_n]})$ in $L_1(J)$, and hence $x_j\chi_{[p_n,2p_n]}\longrightarrow x_0\chi_{[p_n,2p_n]}$ in $L_1(J)$, from which it follows that $x_j\chi_{[0,2p_n]}\longrightarrow x_0\chi_{[0,2p_n]}$ in $L_1(J)$ when $j\to\infty$. By repeating this argument we find $\lim_{j\to\infty} x_j\chi_{[0,kp_n]}=x_0\chi_{[0,kp_n]}$ in $L_1(J)$ for $k=1,\cdots,n-1$, i.e. $\lim_{j\to\infty} x_j=x_0$ in $L_1(J)$. This proves the continuity of T_n^{-1}. Consequently, T_n is a homemorphism $L_1(J)\longrightarrow L_1(J)$.

Since $G(L_1(J))\subset H(B_\varphi^m(J))$ and $H(B_\varphi^m(J))$ is compact in $L_1(J)$, G is a compact mapping, and therefore $T=I-G$ is a proper mapping. Thus we can apply Browder's theorem [1; Th. 7], which proves that $T^{-1}(0)$ is a compact R_δ in $L_1(J)$. Since $x(t)=G(x)(t)\in K$ for $t\in J$ and $x\in T^{-1}(0)$, $r(x(t))=x(t)$, and finally $T^{-1}(0)=V$.

Remark 1. Let $S(J)$ be the space of all L-measurable functious $u\colon J\longrightarrow$

R^d. Assume that $F(J)$ is a Frechet function space with paranorm $|\cdot|_F$ such that

1° $L^\infty(J)\subset F(J)\subset S(J)$.

2° If $u_n, u\in F(J)$ and $\lim_{n\to\infty}|u_n-u|_F=0$, then $u_n\to u$ in $S(J)$.

3° If $u_n, u\in S(J)$, $k\in R^+$, $\lim_{n\to\infty}u_n(s)=u(s)$ and $|u_n(s)|\leq k$, $|u(s)|\leq k$ for almost every $s\in J$, then $\lim_{n\to\infty}|u_n-u|_F=0$.

For each $x, y\in V$ put $d_1(x, y)=\|x-y\|_1$ and $d_F(x, y)=|x-y|_F$. Since V is a set of continuous functions $x: J\longrightarrow R^d$ such that $|x(s)|\leq|f(s)|+h$ for $s\in J$, the metric spaces $\langle V, d_1\rangle, \langle V, d_F\rangle$ are homeomorphic. This proves that the set V of all solutions of (1) defined on J is a continuum in $F(J)$.

Remark 2. Let $C(J)$ be the space of all continuous functions $u: J\longrightarrow R^d$ with the norm $\|u\|_C=\sup\{|u(t)|: t\in J\}$. Replacing the condition (iii, 2°) by a stronger condition:

"the mapping $I\longrightarrow L_\varphi^*(J)^d$ defined by $t\longrightarrow a(t, \cdot)$ is continuous in the strong topology on $L_\varphi^*(J)^d$",

we see that V is an equicontinuous bounded subset of $C(J)$, since

$$|x(t)-x(\tau)|=\left|\int_0^t a(t, s)g(s, x(s))ds-\int_0^\tau a(\tau, s)g(s, x(s))ds\right|$$
$$\leq\left|\int_\tau^t a(t, s)g(s, x(s))ds\right|+\left|\int_0^\tau (a(t, s)-a(\tau, s))g(s, x(s))ds\right|$$
$$\leq\sup\{\|a(t, \cdot)\|_\phi: t\in J\}\|m\chi_{[\tau, t]}\|_\varphi+\|m\|_\varphi\|a(t, \cdot)-a(\tau, \cdot)\|_\phi$$

for each $x\in V$, $t, \tau\in J$, and $\|m\chi_{[\tau, t]}\|_\varphi\to 0$, $\|a(t, \cdot)-a(\tau, \cdot)\|_\phi\to 0$ when $|t-\tau|\to 0$. Consequently, the metric spaces $\langle V, d_1\rangle, \langle V, d_C\rangle$ are homeomorphic, and therefore V is a continuum in $C(J)$.

References

[1] F. E. Browder, C. P. Gupta, Topological degree and nonlinear mappings of analytic type in Banach space, J. Math. Anal. Appl., **26** (1969), 390-402.

[2] N. Kikuchi, S. Nakagiri, An existence theorem of solutions of nonlinear integral equations, Funkcial. Ekvac., **15** (1972), no. 2, 131-138.

[3] M. A. Krasnoselskii, Ja. B. Rutickii, Vypuklye funktcii i prostranstva Orlicza, Moskva, 1958.

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