Funkcialaj Ekvacioj, **17** (1974), 25-29

On Start Sets and Negative Prolongational Limit Sets in Semidynamical Systems

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Abstract. Negative prolongational limit sets are introduced. It is shown that if X is Hausdorff rim-compact, then $J^{-}(x)$ compact and non-empty implies $\lambda^{-}(x)\neq\phi$. Also if S is closed, then $J^{-}(x)$ is weakly negatively invariant. An example is given to show that the above conclusion need not follow if S is not closed. An apparently new necessary and sufficient condition for a point x to be a start point is included (proposition 2.1). In addition, some of its consequences are given as corollaries.

1. Introduction.

Let X be a Hausdorff space and R^{+} the set of non-negative real numbers. A *semidynamical system* on X is the triple (X, π, R^{+}) where π is a mapping from $X\times R^{+}$ to X satisfying the following conditions:

1. $x\pi 0=x$ for all $x\in X$.
2. $(x\pi t)\pi s=x\pi(t+s)$ for all $x\in X$ and $t, s\in R^{+}$.
3. π is continuous.

A point $x\in X$ is called a *start point* if $x\neq y\pi t$ for all $y\in X$ and $t>0$. The set S of start points is called the *start set*. For $x\in X-S(\equiv C(S))$, let θ_{x} be a negative semitrajectory (see [2]). Then θ_{x} is one of following types:

(I) there is a $y\in S$ and $t>0$ such that $\theta_{x}=y\pi[0, t]$;

(II) whatever $t>0$ there is a $y\in\theta_{x}$ such that $x=y\pi t$ (*principal*);

(III) $\theta_{x}\cap S=\phi$ and $\sup\{t>0 \mid \text{for some } y\in\theta_{x}, y\pi t=x\}<+\infty$.
(*non-principal, non-compact*).

Remark. We note that if θ_{x} is of type III, it cannot be contained in K, a compact subset of X.

For $x\in X$, $F(x)$ the negative funnel through x, is defined as follows:

$$F(x)=\{y \mid y\pi t=x \quad \text{for some } y\in X \text{ and } t\in R^{+}\}.$$

For $U\subset X$, let $F(U)$ denote the set $\bigcup_{x\in U} F(x)$,

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* Research supported by financial assistance from the Council of Scientific and Industrial Research, Government of India.

In addition, we shall use the following notation:

$$T(\theta_x)=\sup\{t\geq 0 \mid \text{ for some } y\in\theta_x, y\pi t=x\}.$$

$T_x=\inf\{T(\theta_x) \mid \theta_x$ is a negative semitrajectory through $x\}$ and $T(U)=\sup\{T(\theta_x) \mid \theta_x\subset F(x)$ for some $x\in U\}$.

We recall that X is called *rim-compact* if for each $x\in X$ there is a neighbourhood U of x such that ∂U is compact. It is easy to show that if X is Hausdorff and rim-compact then X is regular (see [3] p. 42). Throughout the paper X stands for a rim-compact, Hausdorff space. Also we recall the following:

$\lambda^+(x)=\{y \mid$ there is a net $\{t_i\}$ (in R^+) diverging to $+\infty$ such that the net $\{x\pi t_i\}$ converges to $y\}$,

$J^+(x)=\{y \mid$ there are nets $\{x_i\}$ converging to x, $\{t_i\}$ (in R^+) diverging to $+\infty$ such that $\{x_i\pi t_i\}$ converges to $y\}$.

2. Start Sets.

We thank the referee for suggesting the following proposition as the basic result leading to our earlier propositions (corollaries 2.1 and 2.2).

Proposition 2.1. *Given an lsd-system on a rim-compact space X, a point $x\in X$ is a start point if and only if $\alpha_y\to 0$ as $y\to x$ where for any $y\in X$, $\alpha_y=-\sup\{t\geq 0 \mid y\in X\pi t\}$.*

Proof. Let x be a start point. Let $\{y_i\}$ be a net converging to x such that the net $\{\alpha_{y_i}\}$ does not converge to 0, that is, there exists an $\varepsilon>0$. such that $-\alpha_{x_i}>\varepsilon$ for all i. Let $z_i\in X$ be such that $z_i\pi\varepsilon=t_i$. Note that the net $\{z_i\}$ cannot have a subnet converging to x. Therefore there is a neighourhood U of $x-\partial U$ may be assumed to be compact-such that the net $\{z_i\}\subset C(U)$ ultimately. Thus there exists $t_i\in[0,\varepsilon]$ with $z_i\pi t_i(=z_i')\in\partial U$. Let $\{z_j'\}$ and $\{t_j\}$ be subnets of $\{z_i'\}$ and $\{t_i\}$ converging to some $z'\in\partial U$ and $\bar{t}\in[0,\varepsilon]$ respectively. Then necessarily $\bar{t}<\varepsilon$ and $z'\pi(\varepsilon-\bar{t})=x$, a contradiction. The other half is obvious.

The following corollaries are immediate:

Corollary 2.1. *Let $x\in S$ and let U be a neighbourhood of x. Then there exists a neihbourhood V (of x) such that $F(V)\subset U$*

Corollary 2.2. *Let $x\in S$. Then there exists a neighbourhood V of x such that $T(V)<+\infty$.*

3. Negative Prolongational Limit Sets.

The importance of the notion of positive prolongation, $D^+(x)$, in the study of stability was recognized by Ura [4]. The positive prolongational limit set, $J^+(x)$, introduced essentially by Auslander, Bhatia and Seibert [1], together with the positive trajectory $\nu^+(x)$ make up $D^+(x)$. The corresponding sets $D^-(x)$, $J^-(x)$ have also been studied in relation to dynamical systems. Let (X, π, R^+) be a semidynamical system.

Definition. $J^-(x)=\{y \mid x\in J^+(y)\}$.

$\lambda^-(x)=\cup\{\lambda^-(\theta_x) \mid \theta_x$ is a principal negative semitrajectory through $x\}$, where $\lambda^-(\theta_x)$ is as in [2], DEF. 11.

Remark. $y\in J^-(x)$ if and only if there exist nets $\{y_i\}$ converging to y in X, $\{t_i\}$ (in R^+) diverging to $+\infty$ such that $\{y_i\pi t_i\}$ converges to x in X.

Proposition 3.1. *Foy any $x\in X$, $J^-(x)$ is a closed positively invariant set.*

Proof. Let $\{y_i\}$ be a net in $J^-(x)$ converging to y. Then $x\in J^+(y_i)$ and thus $x\in J^+(y)$. Hence $J^-(x)$ is closed. Moreover, $x\in J^+(y)\subset J^+(y\pi t)$ for $t>0$ implies $y\pi t\in J^-(x)$. Hence $J^-(x)$ is positively invariant.

The next proposition is an improved version of a result in an earlier draft.

Proposition 3.2. *Let $x\in X$ be such that $J^-(x)$ is compact and nonempty. Then $\lambda^-(x)$ is non-empty.*

Proof. Suppose $\lambda^-(x)$ is empty. If $y(\in X)$ is such that $x\in J^+(y)$, then there exists a negative semitrajectory θ_x (through x) of type III or of principal type such that $\theta_x\subset J^+(y)$. Since $\lambda^-(\theta_x)$ is empty, θ_x is a closed set.

Let $T\geq 0$ be such that for any $t\geq T, z\in\theta_x$ with $z\pi t=x$ implies $z\in J^-(x)$. Let $N=\{z\in\theta_x \mid z\pi t=x$ and $t\geq T\}$. Then N is a closed. There exists a closed neighbourhood V of N and an open neighbourhood U of $J^-(x)$ such that ∂U is compact and $\bar{U}\cap V$ is empty. Let $z\in\theta_x$ be arbitrary with $z\pi t_z=x$, where $t_z\geq T$. Then $y\in J^-(z)$ implies that there exist nets $\{y_i\}$ $(\subset U)$ converging to y and $\{t_i\}$ in R^+ diverging to $+\infty$ such that $y_i\pi t_i=z_i$ and the net $\{z_i\}$ converging to z. Let $T_i\in[0, t_i]$ be such that $y_i\pi T_i\in\partial U$ and also $y_i\pi[0, T_i)\subset U$. Without loss of generality we can assume that the net $\{y_i\pi t_i\}$ converges to a point $\bar{y}\in\partial U$. If $\{T_i\}$ has a subnet conveging to say $t^*\in R^+$, then $\bar{y}=y\pi t^*$, whence $\bar{y}\in J^-(x)$, a contradiction. Hence the net $\{T_i\}$ diverges to $+\infty$. A similar contradiction is arrived at if the net $\{(t_i-T_i)\}$ is divergent. Thus we may suppose that $\{(t_i-T_i)\}$ has a subnet converging to $\bar{t}$ and $\bar{y}\pi\bar{t}=z$. Let $\{z_i\}$ be a net in θ_x with $z_i\pi t_{z_i}=x$, and let the net $\{t_{z_i}\}$ converge to $T(\theta_x)$. Let the corresponding net $\{\bar{y}_i\}$ $(\subset\partial U)$ have a subnet $\{\bar{y}_j\}$ converging to some $y^*\in\partial U$. The net $\{z_j\}$ cannot have a convergent subnet for then in case $T(\theta_x)<+\infty$, $z\in\theta_x$ and $z\pi T(\theta_x)=x$; otherwise $\lambda^-(\theta_x)\neq\phi$. Thus the net $\{\bar{t}_i\}$ and therefore the net $\{\bar{t}_j+t_{z_j}\}$ is unbounded. Hence $\bar{z}\in J^-(x)$, a contradiction. This completes the proof.

Proposition 3.3. *Let $S(\subset X)$ be closed. Then for any $x\in X$, $J^-(x)$ is weakly negatively invariant.*

Proof. Let $y\in\partial(J^-(x))-S$. For weak negative invariance it is sufficient to prove that there exist a $t>0$ and $z\in J^-(x)$ such that $z\pi[0,t]\subset J^-(x)$ and $z\pi t=x$. Since S is closed, there is a neighbourhood U of y with ∂U compact and $U\cap S=\phi$. Moreover $y\in J^-(x)$ imples that there exist nets $\{y_i\}$ $(\subset U)$ converging to y, $\{t_i\}$ in R^+ diverging to $+\infty$ such that $\{y_i\pi t_i\}$ converges to x. For each i, there is z_i such that either we have $z_i\pi 1=y_i$ or $z_i\pi T_i=y_i$, $T_i\in[0,1)$ and $z\in S$. If for every neigbourhood V of y the net $\{z_i\}$ has a subnet in V, then we can find nets $\{z_{i(V)}\}$ and $\{y_{i(V)}\}$ such that $z_{i(V)}$ as well as $y_{i(V)}\in V$. Thus from $\{t_{i(V)}-1\}$ diverging to $+\infty$ and both the above nets obviously converging to y, we get $y\pi[0,1]\subset J^-(x)$ and $y\pi 1=y$. Otherwise, there is a neighbourhood $V(\subset U)$ of y with $z_i\in V$. Let $t_i(\in[0,1])$ be such that $z_i\pi t_i\in \partial V$. The net $\{\tau_i\}$, where

$$\tau_i=\begin{cases} T_i-t_i & \text{if } z_i\in S,\\ 1-t_i & \text{otherwise,}\end{cases}$$

is bounded. Let $\{z_j\pi t_j\}$ and $\{\tau_j\}$ be subnets of the above nets converging to $z\in\partial V$ and $t\in(0,1]$ respectively. Then $z\pi[0,t]\subset J^-(x)$ and $z\pi t=y$. This completes the proof.

The following example shows that if S is not colosed, then $J^-(x)$ need *not* be weakly negatively invariant.

Example.

Let $(R^2\supset)X=\{(x,y)\mid x=0 \text{ or } y=0\}\cup\{(x,y)\mid x\in[-1,0]\ y\in(0,\infty) \text{ and } -1\leq xy<0\}$.

Let the dynamical system on R^2 determined by the system

$$\frac{dx}{dt}=-x,\quad \frac{dy}{dt}=y,$$

be restricted to X. Then, the start point set S is $\{(x,y)\mid x=-1 \text{ and } y\in(0,1]\}$. Obviously, S is not closed. Moreover, for any point $(0,y)\in X$, $(-1,0)\in J^-(0,y))$; whereas any point $(x,0)$, $x<-1$, belongs to $F((-1,0))$ but does not belong to $((0,y))$. Hence, $J^-((0,y))$ is *not* weakly negatively invariant.

References

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(Ricevita la 6-an de aŭgusto 1973)
(Reviziita la 17-an de novembro, 1973)