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On the Hypertranscendency of Solutions of a Difference Equation of Kimura

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§ 1. Introduction.

It is well known that the gamma function $\Gamma(x)$ which is a solution of the difference equation

$$y(x+1)=xy(x) \tag{1.1}$$

does not satisfy any algebraic differential equation (Hölder, [4]). By an algebraic differential equation we mean a differential equation of the form

$$F(x, y, y', \cdots, y^{(n)})=0$$

where F is a polynomial in $y, y', \cdots, y^{(n)}$ whose coefficients are rational functions of x. We denote by $\boldsymbol{C}(x)$ the set of all rational functions of x over the complex number field $\boldsymbol{C}$. Then $\boldsymbol{C}(x)$ is a differential field in a natural way and the polynomial F can be regarded as an element of the differential ring extension $\boldsymbol{C}(x)\{y\}$ of $\boldsymbol{C}(x)$, where y is a differential indeterminate.

We shall say that a function is hypertranscendental over the differential field $\boldsymbol{C}(x)$ if it does not satisfy any algebraic differential equation. The above famous theorem of Hölder is then expressed by saying that $\Gamma(x)$ is hypertranscendental over the differential field $\boldsymbol{C}(x)$. A proof of this theorem shows that every non-trivial solution of equation (1.1) is hypertranscendental over $\boldsymbol{C}(x)$.

Many interesting facts analogous to the above were obtained for equations of the form

$$y(x+1)=\frac{a(x)y(x)+b(x)}{c(x)y(x)+d(x)},$$
$$y(qx)=\frac{a(x)y(x)+b(x)}{c(x)y(x)+d(x)},$$

by H. Tietze [10], E. Stridsberg [9] and T. E. Mason [6], where a, b, c and d are in $\boldsymbol{C}(x)$ and q is a constant. Another interesting example of a hypertranscendental function is the Riemann zeta function, $\zeta(x)$ (Hilbert, [3]).

We like to remark the following. The concept of hypertranscendency over a differential field can be naturally transferred to that over a difference field. Observing that $\boldsymbol{C}(x)$ is also a difference field, we can define the algebraic difference equation in the same way. We say that a function is hypertranscendental over the difference field $\boldsymbol{C}(x)$ if it does not satisfy any algebraic difference equation. K. Okubo proved in his unpublished paper that the function $\log x$, which is a solution of

$$xy'-1=0,$$

is hypertranscendental over the difference field $\boldsymbol{C}(x)$.

Recently, T. Kimura [5] has studied a nonlinear difference equation

$$y(x+1)=y(x)+1+\frac{\lambda}{y(x)}, \tag{1.2}$$

where λ is a nonzero constant, and he obtained a very interesting solution meromorphic in $|x|<\infty$. In his paper, he conjectured that the solution, or more generally, any nontrivial solution of (1.2) is hypertranscendental over the differential field $\boldsymbol{C}(x)$, where the only trivial solution of (1.2) is the constant solution $y(x)\equiv-\lambda$.

The purpose of this paper is to prove that his conjecture is true, namely, to prove the

Theorem 1. *Any nontrivial solution of* (1.2) *is hypertranscendental over the differential field* $\boldsymbol{C}(x)$.

It should be noted that equation (1.2) may have multiplevalued solutions.

Let $\varphi(x)$ be a nontrivial solution of (1.2). Note that the mapping φ has its inverse mapping ψ. The mapping φ forms the following commutative diagram together with $f(z)=z+1+(\lambda/z)$ and $g(z)=z+1$:

$$\begin{array}{ccc} \cdot & \xrightarrow{\ f\ } & \cdot \\ \varphi\big\uparrow & & \big\uparrow\varphi \\ \cdot & \xrightarrow[\ g\ ]{} & \cdot \end{array},$$

so ψ satisfies the commutative diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{\ f\ } & \cdot \\ \psi\big\downarrow & & \big\downarrow\psi \\ \cdot & \xrightarrow[\ g\ ]{} & \cdot \end{array}.$$

This means that $\psi(x)$ is a solution of the functional equation

$$y\left(x+1+\frac{\lambda}{x}\right)=y(x)+1. \tag{1.3}$$

It is clear that equation (1.3) has no constant solution and that the hypertranscendency of φ over $\boldsymbol{C}(x)$ is equivalent to that of ψ over $\boldsymbol{C}(x)$. Therefore, Theorem 1 is equivalent to the

Theorem 2. *Any solution of* (1.3) *is hypertranscendental over the differen-*

tial field $\boldsymbol{C}(x)$.

We shall prepare several preliminary propositions in section 2 and shall carry out the proof of Theorem 2.

§2. Preliminary propositions.

Let us set

$$f(x)=x+1+\frac{\lambda}{x},$$

λ being a nonzero constant and denote by $f_n(x)$ the n-th iterate of $f(x)$, where $f_0(x)=x$ and $f_1(x)=f(x)$. For any $\alpha\in\boldsymbol{P}$, we define $A(\alpha)$ to be the set

$$\{x\in\boldsymbol{P};\ f_n(x)=\alpha \quad \text{for some} \quad n>0\},$$

where $\boldsymbol{P}$ is the extended complex plane. We shall first prove the

Proposition 1. *Both* $A(0)$ *and* $A(\infty)$ *are infinite pointsets.*

Proof. From $f(0)=\infty$ and $f(\infty)=\infty$, it follows that

$$A(\infty)=\{0\}\cup A(0),$$

hence we have only to show that $A(0)$ is an infinite set.

To do this, it suffices to verify that for any sequence $\{x_1, x_2, \cdots\}$ such that

$$f(x_1)=0, f(x_2)=x_1, \cdots, f(x_k)=x_{k-1}, \cdots,$$

the x_k's are mutually distinct. Suppose that for some positive integers $l>k$,

$$x_l=x_k.$$

Since

$$f_{l-k}(x_l)=x_k, \quad \text{we have}$$
$$f_{l-k}(x_k)=x_k,$$

from which follows

$$f_k(f_{l-k}(x_k))=f_k(x_k).$$

From the definition of $\{x_k\}_{k=1}^{\infty}$, we have

$$f_k(x_k)=0.$$

On the other hand,

$$f_k(f_{l-k}(x_k))=f_{l-k}(f_k(x_k))=f_{l-k}(0)=\infty.$$

This is a contradiction. The proposition is thus proved.

Proposition 2. *The functional equation*

$$a\left(x+1+\frac{\lambda}{x}\right)=a(x)+c \quad (c\text{: constant and } \lambda\neq 0) \tag{2.1}$$

has a rational function solution if and only if $c=0$. *In the case* $c=0$, *any rational function solution is necessarily a constant function.*

Proof. We shall prove the first assertion. If $c=0$, then (2.1) has rational function solutions. In fact $y(x)\equiv$ constant is a solution.

Suppose that $c\neq 0$ and that (2.1) has a rational function solution $a(x)$. Taking in (2.1) the limit as $x\to\infty$, we have

$$a(\infty)=a(\infty)+c.$$

This equality implies that $x=\infty$ is a pole of $a(x)$. We conclude from this fact that any point $x\in A(\infty)$ is also a pole of $a(x)$. By virtue of Proposition 1, $a(x)$ has infinitely many poles. This contradicts the hypothesis that $a(x)$ is a rational function.

We next show the second assertion. Suppose that $a(x)$ is a nonconstant rational function satisfying

$$a\left(x+1+\frac{\lambda}{x}\right)=a(x). \tag{2.2}$$

It is easily verified by the same reason as above that $x=\infty$ is neither a pole nor a zero of $a(x)$ and furthermore that $x=0$ is neither a pole nor a zero of $a(x)$. Hence, $a(x)$ is represented as

$$a(x)=\frac{c(x)}{d(x)},$$

where $c(x)$ and $d(x)$ are relatively prime polynomials of the same degree $n>0$:

$$c(x)=\sum_{j=0}^{n}c_jx^j,\quad c_0, c_n\neq 0,$$
$$d(x)=\sum_{j=0}^{n}d_jx^j,\quad d_0, d_n\neq 0.$$

From (2.2) we get

$$\frac{x^nc\left(\dfrac{x^2+x+\lambda}{x}\right)}{x^nd\left(\dfrac{x^2+x+\lambda}{x}\right)}=\frac{c(x)}{d(x)}. \tag{2.3}$$

Observing that

$$x^nc\left(\frac{x^2+x+\lambda}{x}\right)\quad\text{and}\quad x^nd\left(\frac{x^2+x+\lambda}{x}\right)$$

are polynomials of degree $2n$ with nonzero constant terms, we see by (2.3) that

$$x^nc\left(\frac{x^2+x+\lambda}{x}\right)=0\quad\text{and}\quad x^nd\left(\frac{x^2+x+\lambda}{x}\right)=0$$

have n common roots $\neq 0$. This is a contradiction because $c(x)$ and $d(x)$ are relatively prime. The proposition is thus completed.

Proposition 3. *Any rational function solution of the equation*

$$\left(\frac{x^2-\lambda}{x^2}\right)^h a\left(x+1+\frac{\lambda}{x}\right)=a(x), \quad (\lambda\neq 0) \tag{2.4}$$

is identically zero, where h is a positive integer.

Proof. Suppose that $a(x)$ is a rational function solution which does not identically vanish. We shall show that $x=\infty$ is neither a pole nor a zero of $a(x)$. For this purpose, we develop $a(x)$ into the Laurent expansion at $x=\infty$:

$$a(x)=\sum_{j=r}^{\infty} a_j x^{-j}, \quad a_r\neq 0.$$

Since $x=\infty$ is a fixed point of

$$f(x)=x+1+\frac{\lambda}{x},$$

we can obtain the Laurent expansion of

$$a\left(x+1+\frac{\lambda}{x}\right)$$

at $x=\infty$, and then that of

$$\left(\frac{x^2-\lambda}{x^2}\right)^h a\left(x+1+\frac{\lambda}{x}\right)$$

immediately from the expansion of $a(x)$ as follows

$$\begin{aligned}
&\left(\frac{x^2-\lambda}{x^2}\right)^h a\left(x+1+\frac{\lambda}{x}\right)\\
&=(1-\lambda h x^{-2}+\cdots)(a_r x^{-r}(1+x^{-1}+\lambda x^{-2})^{-r}+a_{r+1}x^{-(r+1)}(1+x^{-1}+\lambda x^{-2})^{-(r+1)}+\cdots)\\
&=a_r x^{-r}+(a_{r+1}-ra_r)x^{-(r+1)}+\cdots\cdots.
\end{aligned}$$

Therefore we have

$$a_{r+1}-ra_r=a_{r+1}, \quad (a_r\neq 0),$$

from which follows

$$r=0.$$

This shows that $x=\infty$ is neither a pole nor a zero of $a(x)$.

Now, by taking the limit in (2.4) as $x\longrightarrow 0$, we have

$$\left(-\frac{x^2}{\lambda}\right)^h a(x)\rightarrow a(\infty), \quad (x\rightarrow 0).$$

Hence, $x=0$ is a pole of $a(x)$ of order $2h$.

We shall next show that $a(x)$ has infinitely many poles. Let us choose a sequence of points $x_0, x_1, \cdots, x_k, \cdots, (x_0=0)$ so that

$$f(x_1)=x_0, f(x_2)=x_1, \cdots, f(x_k)=x_{k-1}, \cdots.$$

As was shown in the proof of Proposition 1, we have

$$x_k \neq x_{k'}, \quad \text{if} \quad k \neq k'.$$

Suppose that $x=x_{k-1}$ is a pole of order l. Then, $a(x)$ is written as

$$a(x)=\frac{\hat{a}(x)}{(x-x_{k-1})^l} \tag{2.5}$$

in a neighbourhood U_{k-1} of $x=x_{k-1}$. Using (2.5) and (2.4), we shall examine the local behavior of $a(x)$ at $x=x_k$. It is easy to see that

$$x+1+\frac{\lambda}{x}-x_{k-1}=\frac{x_k^2-\lambda}{x_k^2}(x-x_k)+\frac{\lambda}{x_k^3}(x-x_k)^2+O((x-x_k)^3),$$

in a neighbourhood U_k of $x=x_k$. Then, from (2.4) and (2.5), we have in U_k

$$a(x)=\left(\frac{x^2-\lambda}{x^2}\right)^h\left(\frac{x_k^2}{x_k^2-\lambda}\right)^l\frac{\hat{a}\left(x+1+\frac{\lambda}{x}\right)}{(x-x_k)^l(1+O(x-x_k))}$$

if $x_k^2-\lambda\neq 0$, and

$$a(x)=\left(\frac{x^2-\lambda}{x^2}\right)^h\left(\frac{x_k^3}{\lambda}\right)^{2l}\frac{\hat{a}\left(x+1+\frac{\lambda}{x}\right)}{(x-x_k)^{2l}(1+O(x-x_k))}$$

if $x_k^2-\lambda=0$. Therefore, $x=x_k$ is also a pole of order at least $m=\text{Min}(l, 2l-h)$.

From this fact and the fact that $x=x_0$ is a pole of order $2h$, it follows that $x_k, k=1, 2, \cdots$, are all poles of $a(x)$. Therefore $a(x)$ cannot be a rational function, which proves Proposition 3.

The last proposition is conscerned with the following equation

$$\frac{x^2-\lambda}{x^2}a\left(x+1+\frac{\lambda}{x}\right)+q(x)=a(x). \tag{2.6}$$

Proposition 4. *Suppose that* (i) $q(x)$ *is a nonzero rational function* (ii) $x=0$ *is the only pole of* $q(x)$ (iii) *the order of the pole* $x=0$ *is greater than two* (iv) $x=\infty$ *is a zero point. Then equation* (2.6) *does not have any rational function solution.*

Proof. We denote by m the order of pole of $q(x)$ at $x=0$ and by n the order of zero of $q(x)$ at $x=\infty$.

Suppose that (2.6) has a rational function solution $a(x)$. It is easy to see that $a(x)$ is not a constant. In fact, if $a(x)\equiv a$, then

$$\frac{x^2-\lambda}{x^2}a+q(x)=a. \tag{2.7}$$

By the assumption on $q(x)$, the left hand side of (2.7) has a pole of order $m\geqq 3$ at $x=0$, which is a contradiction.

We see that $x=0$ cannot be a pole of $a(x)$. In fact, if $x=0$ were a pole of $a(x)$, then from supposition (ii) we infer in the same way as in the proof of Proposition 3 that $a(x)$ would have infinitely many poles.

Now, let

$$a(x)=\sum_{j=r}^{\infty}a_jx^{-j},\quad a_r\neq 0 \tag{2.8}$$

be the Laurent expansion of $a(x)$ at $x=\infty$. Since $f(0)=\infty$, the Laurent expansion of $a(x)$ at $x=0$ can be derived from (2.6) and (2.8). Indeed, from

$$a\left(x+1+\frac{\lambda}{x}\right)=\lambda^{-r}a_rx^r(1+O(x))$$
$$q(x)=q'x^{-m}(1+O(x)),\quad q'\neq 0$$

as $x\to 0$, we have

$$\begin{aligned}a(x)&=\frac{x^2-\lambda}{x^2}a\left(x+1+\frac{\lambda}{x}\right)+q(x)\\&=-\lambda^{-r+1}a_rx^{r-2}(1+O(x))+q'x^{-m}(1+O(x)),\end{aligned}$$

as $x\to 0$. Since $x=0$ is not a pole of $a(x)$, we have

$$r-2=-m$$
$$-\lambda^{-r+1}a_r+q'=0.$$

From the first relation and $m\geqq 3$, we get

$$r\leqq -1. \tag{2.9}$$

This means that $x=\infty$ is a pole of $a(x)$ of order at least 1.

Finally, we shall consider the Laurent expansion of the left hand side of (2.6) at $x=\infty$.

Since

$$\frac{x^2-\lambda}{x^2}a\left(x+1+\frac{\lambda}{x}\right)=a_rx^{-r}+(a_{r+1}-ra_r)x^{-(r+1)}+O(x^{-(r+2)})$$

and

$$q(x)=q''x^{-n}(1+O(x^{-1})),\quad q''\neq 0$$

as $x\to\infty$, we have by (2.9) and $n\geqq 1$

$$\frac{x^2-\lambda}{x^2}a\left(x+1+\frac{\lambda}{x}\right)+q(x)=a_rx^{-r}+(a_{r+1}-ra_r)x^{-(r+1)}+\cdots.$$

Comparing this expansion with (2.8), we get

$$a_{r+1} - r a_r = a_{r+1}.$$

This is a contradiction because of $r a_r \neq 0$. Thus we have proved Proposition 4.

§ 3. Proof of Theorem 2.

The differential ring $\boldsymbol{C}(x)\{y\}$ is, by definition, isomorphic to the polynomial ring $\mathfrak{R} = \boldsymbol{C}(x)[y_0, y_1, \cdots]$ in infinitely many usual indeterminates $y_0, y_1, \cdots$ by the correspondence $y \longrightarrow y_0, y' \longrightarrow y_1, \cdots$. Then the set of all algebraic differential equations can be put in a one to one correspondence with $\mathfrak{R}$ in a natural way.

Take an arbitrary solution $y(x)$ of equation (1.3) and keep it fixed. Suppose that $y(x)$ satisfies algebraic differential equations. Let $\mathfrak{M}$ be the subset of $\mathfrak{R}$ which corresponds to the set of algebraic differential equations satisfied by $y(x)$. We want to choose an element of $\mathfrak{M}$ which is simplest in some sense.

For a monomial $a(x) y_0^{k_0} y_1^{k_1} \cdots y_n^{k_n}$, we call the ordered system $(k_0, k_1, \cdots, k_n)$ its index and write it simply as

$$k = (k_0, k'), \quad k' = (k_1, \cdots, k_n).$$

We define $|k'|$ by

$$|k'| = k_1 + 2k_2 + \cdots + n k_n.$$

Let us introduce a linear order in the set of indices as follows. Let $k = (k_0, k')$ and $l = (l_0, l')$ be indices of monomials $a(x) y_0^{k_0} y_1^{k_1} \cdots y_n^{k_n}$ and $b(x) y_0^{l_0} y_1^{l_1} \cdots y_m^{l_m}$. We define that

$$k \succ l$$

if either

(i) $|k'| > |l'|$, or

(ii) $|k'| = |l'|$ and the first nonzero integer of

$$k_p - l_p, k_{p-1} - l_{p-1}, \cdots, k_1 - l_1$$

is positive, where $p = \mathrm{Max}(n, m)$, or

(iii) $k' = l'$ and $k_0 > l_0$.

In (ii) we made the convention, if $n < p$, then $k_{n+1} = \cdots = k_p = 0$ and if $m < p$, then $l_{m+1} = \cdots = l_p = 0$. It is easy to see that this order is a linear one. Given a polynomial in $\mathfrak{R}$, we call the greatest index among indices of the nonzero monomials in the polynomial the rank of the polynomial.

We can now choose from $\mathfrak{M}$ a polynomial $F(x, y_0, y_1, \cdots, y_n)$ of the lowest rank. We write F as

$$(3.1)\qquad \begin{aligned} &F(x, y_0, y_1, \cdots, y_n)\\ &=a_{s_0,s'}(x)y_0^{s_0}y_1^{s_1}\cdots y_n^{s_n}+\sum_{k_0=0}^{s_0-1}a_{k_0,s'}(x)y_0^{k_0}y_1^{s_1}\cdots y_n^{s_n}\\ &+\sum_{\substack{|k'|=|s'|\\ k\prec s}}\sum_{k_0=0}^{m(k')}a_{k_0,k'}(x)y_0^{k_0}y_1^{k_1}\cdots y_n^{k_n}\\ &+\sum_{|k'|<|s'|}\sum_{k_0=0}^{m(k')}a_{k_0,k'}(x)y_0^{k_0}y_1^{k_1}\cdots y_n^{k_n}, \end{aligned}$$

where $a_{s_0,s'}(x)y_0^{s_0}y_1^{s_1}\cdots y_n^{s_n}$ is the monomial of the greatest index. We may assume without loss of generality that

$$(3.2)\qquad a_{s_0,s'}(x)=1.$$

It is clear that the polynomial F is determined uniquely under condition (3.2).

It follows from the definition of F that

$$F(x, y(x), y'(x), \cdots, y^{(n)}(x))=0.$$

Since this relation is an identity in x, we obtain

$$(3.3)\qquad F\left(x+1+\frac{\lambda}{x}, y\left(x+1+\frac{\lambda}{x}\right), \cdots, y^{(n)}\left(x+1+\frac{\lambda}{x}\right)\right)=0.$$

Using equation (1.3), we have

$$(3.4)\qquad y^{(k)}\left(x+1+\frac{\lambda}{x}\right)=\left(\frac{x^2}{x^2-\lambda}\right)^k y^{(k)}(x)+\sum_{l=1}^{k-1}p_{k,l}(x)y^{(l)}(x)\qquad k=1, 2, \cdots, n,$$

where the $p_{k,l}(x)$'s are in $\boldsymbol{C}(x)$, in particular,

$$(3.5)\qquad p_{k,k-1}(x)=\frac{d}{dx}\left(\sum_{l=1}^{k-1}\left(\frac{x^2}{x^2-\lambda}\right)^l\right).$$

We put

$$(3.6)\qquad \begin{aligned} &\hat{F}(x, y_0, y_1, \cdots, y_n)\\ &=\left(\frac{x^2-\lambda}{x^2}\right)^{|s'|}F\left(x+1+\frac{\lambda}{x}, y_0+1, \frac{x^2}{x^2-\lambda}y_1, \left(\frac{x^2}{x^2-\lambda}\right)^2 y_2+p_{2,1}(x)y_1, \cdots\right.\\ &\left.\cdots, \left(\frac{x^2}{x^2-\lambda}\right)^n y^n+\sum_{l=1}^{n-1}p_{n,l}(x)y_l\right). \end{aligned}$$

Identities (3.3) and (3.4) show that $\hat{F}$ is a polynomial in $\mathfrak{M}$ of the same rank as F and moreover that the coefficient of the monomial of the greatest index in $\hat{F}$ is equal to 1. Therefore we have

$$(3.7)\qquad F(x, y_0, y_1, \cdots, y_n)=\hat{F}(x, y_0, y_1, \cdots, y_n).$$

Using (3.7), we shall show that we have

$$F(x, y_0, y_1, \cdots, y_n) = y_1. \tag{3.8}$$

This gives a contradiction. Thus the proof of Theorem 2 will be completed.

The rest of this section will be devoted to the proof of (3.8). We first notice that $\hat{F}$ is of the form

$$\begin{aligned}
\hat{F}(x, y_0, y_1, \cdots, y_n) &= (y_0+1)^{s_0} y_1^{s_1} (y_2+q_{2,1}(x)y_1)^{s_2} \cdots \Big(y_n + \sum_{l=1}^{n-1} q_{n,l}(x) y_l\Big)^{s_n} \\
&+ \sum_{k_0=0}^{s_0-1} a_{k_0,s'}\Big(x+1+\frac{\lambda}{x}\Big)(y_0+1)^{k_0} y_1^{k_1}(y_2+q_{2,1}(x)y_1)^{k_2} \\
&\cdots \Big(y_n + \sum_{l=1}^{n-1} q_{n,l}(x) y_l\Big)^{k_n} \\
&+ \sum_{\substack{|k'|=|s'| \\ k\prec s}} \sum_{k=0}^{m(k')} a_{k_0,k'}\Big(x+1+\frac{\lambda}{x}\Big)(y_0+1)^{k_0} y_1^{k_1}(y_2+q_{2,1}(x)y_1)^{k_2} \\
&\cdots \Big(y_n + \sum_{l=1}^{n-1} q_{n,l}(x) y_l\Big)^{k_n} \\
&+ \sum_{|k'|<|s'|} \sum_{k_0=0}^{m(k')} \Big(\frac{x^2-\lambda}{x^2}\Big)^{|s'|-|k'|} a_{k_0,k'}\Big(x+1+\frac{\lambda}{x}\Big)(y_0+1)^{k_0} y_1^{k_1}(y_2+q_{2,1}(x)y_1)^{k_2} \\
&\cdots \Big(y_n + \sum_{l=1}^{n-1} q_{n,l}(x) y_l\Big)^{k_n},
\end{aligned} \tag{3.9}$$

where

$$q_{k,l}(x) = \Big(\frac{x^2-\lambda}{x^2}\Big)^k p_{k,l}(x). \tag{3.10}$$

Our proof is divided into five steps.

The first step is to show that $s_0=0$. Suppose to the contrary that $s_0>0$. We see by a simple calculation that the coefficient of the term of the index $(s_0-1, s_1, \cdots, s_n)$ in $\hat{F}$ is

$$a_{s_0-1,s'}\Big(x+1+\frac{\lambda}{x}\Big)+s_0.$$

From (3.7) we get

$$a_{s_0-1,s'}\Big(x+1+\frac{\lambda}{x}\Big)+s_0 = a_{s_0-1,s'}(x).$$

This contradicts Proposition 2. Thus F can be written as

$$\begin{aligned}
F(x, y_0, y_1, \cdots, y_n) &= y_1^{s_1} \cdots y_n^{s_n} \\
&+ \sum_{\substack{|k'|=|s'| \\ k\prec s}} \sum_{k_0=0}^{m(k')} a_{k_0,k'}(x) y_0^{k_0} y_1^{k_1} \cdots y_n^{k_n}
\end{aligned}$$

$$+\sum_{|k'|<|s'|}\sum_{k_0=0}^{m(k')} a_{k_0,k'}(x) y_0^{k_0} y_1^{k_1}\cdots y_n^{k_n}.$$

The second step is to prove that $m(k')=0$ and $a_{0,k'}(x)=a_{k'}$ (constant) for any k' with $|k'|=|s'|$, $k\prec s$. Suppose that $m(k')>0$ for some k' with $|k'|=|s'|$, $k\prec s$. Observing that the coefficients of the terms of the indices $(m(k'),k')$ and $(m(k')-1,k')$ are

$$a_{m(k'),k'}\left(x+1+\frac{\lambda}{x}\right) \quad\text{and}\quad a_{m(k')-1,k'}\left(x+1+\frac{\lambda}{x}\right)+m(k')a_{m(k'),k'}\left(x+1+\frac{\lambda}{x}\right)$$

respectively, we have from (3.7)

$$a_{m(k'),k'}\left(x+1+\frac{\lambda}{x}\right)=a_{m(k'),k'}(x),$$

$$a_{m(k')-1,k'}\left(x+1+\frac{\lambda}{x}\right)+m(k')a_{m(k'),k'}\left(x+1+\frac{\lambda}{x}\right)=a_{m(k')-1,k'}(x).$$

By Proposition 2 and the first equality above, we have

$$a_{m(k'),k'}(x)=a_{m(k'),k'}=\text{constant}.$$

Then, by applying the same proposition to the second equation, we get

$$m(k')=0.$$

Hence, F is written as

$$F(x, y_0, y_1, \cdots, y_n)$$
$$=y_1^{s_1}\cdots y_n^{s_n}+\sum_{\substack{|k'|=|s'|\\ k\prec s}} a_{k'} y_1^{k_1}\cdots y_n^{k_n}+\sum_{|k'|<|s'|}\sum_{k_0=0}^{m(k')} a_{k_0,k'}(x) y_0^{k_0} y_1^{k_1}\cdots y_n^{k_n}.$$

The third step is to see that $m(k')=0$ for any $|k'|<|s'|$. Suppose that $m(k')>0$ for some k' with $|k'|<|s'|$. Let h' be the greatest among such indices. Since the coefficient of the term of the index $(m(h'),h')$ in $\hat{F}$ is

$$\left(\frac{x^2-\lambda}{x^2}\right)^{|s'|-|h'|} a_{m(h'),h'}\left(x+1+\frac{\lambda}{x}\right),$$

it follows from (3.7) that

$$\left(\frac{x^2-\lambda}{x^2}\right)^{|s'|-|h'|} a_{m(h'),h'}\left(x+1+\frac{\lambda}{x}\right)=a_{m(h'),h'}(x).$$

By Proposition 3, we get

$$a_{m(h'),h'}(x)\equiv 0,$$

which proves the assertion in this step. Thus F is written as

$$F(x, y_0, y_1, \cdots, y_n)=y_1^{s_1}\cdots y_n^{s_n}+\sum_{\substack{|k'|=|s'|\\ k<s}} a_{k'}y_1^{k_1}\cdots y_n^{k_n}+\sum_{|k'|<|s'|} a_{k'}(x)y_1^{k_1}\cdots y_n^{k_n}.$$

The fourth step is to show that $s_2=s_3=\cdots=s_n=0$. Suppose that $s_j>0$ for some $j\geqq 2$. Then the highest monomial of

$$y_1^{s_1}(y_2+q_{2,1}y_1)^{s_2}\cdots\left(y_n+\sum_{l=1}^{n-1} q_{n,l}\,y_l\right)^{s_n}-y_1^{s_1}y_2^{s_2}\cdots y_n^{s_n}$$

is $s_r q_{r,r-1}y_1^{s_1}\cdots y_{r-2}^{s_{r-2}}y_{r-1}^{s_{r-1}+1}y_r^{s_r-1}y_{r+1}^{s_{r+1}}\cdots y_n^{s_n}$ where $r=\mathrm{Min}\{j;\ s_j>0, j\geqq 2\}$. We denote by T_j, $j\geqq 2$ the mapping defined by

$$(k_1, k_2, \cdots)\longrightarrow(k_1, \cdots, k_{j-2}, k_{j-1}+1, k_j-1, k_{j+1}, \cdots).$$

We may notice that if both $(*, k')$ and $(*, T_j(k'))$ are indices then

$$|T_j(k')|=|k'|-1.$$

We shall find the coefficient of the term of the order $(0, \sigma')$, $\sigma'=T_r(s')$, in $\hat{F}$,

$$\begin{aligned}(3.11)\qquad \hat{F}=&\,y_1^{s_1}(y_2+q_{2,1})^{s_2}\cdots\left(y_n+\sum_1^{n-1} q_{n,l}\,y_l\right)^{s_n}\\ &+\sum_{\substack{|k'|=|s'|\\ k<s}} a_{k'}y_1^{k_1}(y_2+q_{2,1})^{k_2}\cdots(y_n+\sum q_{n,l}\,y_l)^{k_n}\\ &+\sum_{|k'|<|s'|}\left(\frac{x^2-\lambda}{x^2}\right)^{|s'|-|k'|}a_{k'}\left(x+1+\frac{\lambda}{x}\right)y_1^{k_1}\cdots(y_n+\sum q_{n,l}\,y_l)^{k_n}.\end{aligned}$$

The coefficients of the terms of that order in the first polynomial and the third one in (3.11) are $s_r q_{r,r-1}$ and

$$\frac{x^2-\lambda}{x^2}a_{\sigma'}\left(x+1+\frac{\lambda}{x}\right)$$

respectively. The coefficient of the term in the second polynomial in (3.11) is $\sum_{T_r(s')=T_j(k')} a_{k'}\cdot k_j\cdot q_{j,j-1}(x)$. Since $r=\mathrm{Min}\{j, s_j>0,\ j\geqq 2\}$ and $k<s$, $|k'|=|s'|$, the relation

$$T_r(s')=T_j(k')$$

is satisfied only for $r\geqq 3$, $j=2$, $k'=T_2^{-1}(\sigma')$. Therefore the coefficient of the term of the order $(0, \sigma')$ in $\hat{F}$ is

$$\frac{x^2-\lambda}{x^2}a_{\sigma'}\left(x+1+\frac{\lambda}{x}\right)+q_r(x),$$

where

$$(3.12)\qquad q_r(x)=\begin{cases} s_r q_{r,r-1}(x)+a_{T_2^{-1}(\sigma')}\cdot q_{2,1}(x), & \text{if } r\geqq 3\\ s_2 q_{2,1}(x), & \text{if } r=2.\end{cases}$$

Here, $q_{j,j-1}(x)$ is a rational function defined by (3.5) and (3.10), thus it is witten by

$$q_{j,j-1}(x)=-\frac{2\lambda}{x^3}\sum_{l=0}^{j-2}(j-l-1)\left(\frac{x^2-\lambda}{x^2}\right)^l. \tag{3.13}$$

From (3.7) we have

$$\frac{x^2-\lambda}{x^2}a_{\sigma'}\left(x+1+\frac{\lambda}{x}\right)+q_r(x)=a_{\sigma'}(x).$$

By (3.12) and (3.13) it can be easily verified that $q_r(x)$ is a nonzero rational function. Furthermore it has the only pole at $x=0$ of order $2r-1(>2)$ and the zero point at $x=\infty$. Then by virtue of Proposition 4, $a_{\sigma'}(x)$ cannot be a rational function. Therefore $s_2=s_3=\cdots=s_n=0$. Thus F is written as

$$F(x, y_0, y_1, \cdots, y_n)=y_1^{s_1}+\sum_{|k'|<s_1}a_{k'}(x)y_1^{k_1}\cdots y_n^{k_n}.$$

The final step is to show that $a_{k'}(x)=0$ for any $|k'|<|s'|$. Let $y_1^{h_1}\cdots y_n^{h_n}$ be the greatest term in $\sum_{|k'|<|s'|}a_{k'}(x)y_1^{k_1}\cdots y_n^{k_n}$. Then the coefficient of $y_1^{h_1}\cdots y_n^{h_n}$ in $\hat{F}$ is

$$\left(\frac{x^2-\lambda}{x^2}\right)^{s_1-|h'|}a_{h'}\left(x+1+\frac{\lambda}{x}\right).$$

Therefore we obtain from (3.7)

$$\left(\frac{x^2-\lambda}{x^2}\right)^{s_1-|h'|}a_{h'}\left(x+1+\frac{\lambda}{x}\right)=a_{h'}(x).$$

By virtue of Proposition 3, it follows from the relation that

$$a_{h'}(x)\equiv 0,$$

which proves that $F=y_1^{s_1}$.

Since F is, by definition, the lowest polynomial in $\mathfrak{M}$, s_1 is necessarily equal to 1. Thus the proof of Theorem 2 has been completed.

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