On the Hypertranscendency of Solutions of a Difference Equation of Kimura

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§ 1. Introduction.

It is well known that the gamma function $\Gamma(x)$ which is a solution of the difference equation

$$ y(x+1) = xy(x) $$

(1.1)

does not satisfy any algebraic differential equation (Hölder, [4]). By an algebraic differential equation we mean a differential equation of the form

$$ F(x, y, y', \cdots, y^{(n)}) = 0 $$

where $F$ is a polynomial in $y, y', \cdots, y^{(n)}$ whose coefficients are rational functions of $x$. We denote by $C(x)$ the set of all rational functions of $x$ over the complex number field $C$. Then $C(x)$ is a differential field in a natural way and the polynomial $F$ can be regarded as an element of the differential ring extension $C(x)[y]$ of $C(x)$, where $y$ is a differential indeterminate.

We shall say that a function is hypertranscendental over the differential field $C(x)$ if it does not satisfy any algebraic differential equation. The above famous theorem of Hölder is then expressed by saying that $\Gamma(x)$ is hypertranscendental over the differential field $C(x)$. A proof of this theorem shows that every non-trivial solution of equation (1.1) is hypertranscendental over $C(x)$.

Many interesting facts analogous to the above were obtained for equations of the form

$$ y(x+1) = \frac{a(x)y(x) + b(x)}{c(x)y(x) + d(x)}, $$

$$ y(qx) = \frac{a(x)y(x) + b(x)}{c(x)y(x) + d(x)}, $$

by H. Tietze [10], E. Stridsberg [9] and T. E. Mason [6], where $a, b, c$ and $d$ are in $C(x)$ and $q$ is a constant. Another interesting example of a hypertranscendental function is the Riemann zeta function, $\zeta(x)$ (Hilbert, [3]).

We like to remark the following. The concept of hypertranscendencey over a differential field can be naturally transferred to that over a difference field. Observing that $C(x)$ is also a difference field, we can define the algebraic difference equation in the same way. We say that a function is hypertranscend-
dental over the difference field $\mathbb{C}(x)$ if it does not satisfy any algebraic difference equation. K. Okubo proved in his unpublished paper that the function $\log x$, which is a solution of

$$xy'-1=0,$$

is hypertranscendental over the difference field $\mathbb{C}(x)$.

Recently, T. Kimura [5] has studied a nonlinear difference equation

$$(1.2) \quad y(x+1)=y(x)+1+\frac{\lambda}{y(x)},$$

where $\lambda$ is a nonzero constant, and he obtained a very interesting solution meromorphic in $|x|<\infty$. In his paper, he conjectured that the solution, or more generally, any nontrivial solution of (1.2) is hypertranscendental over the difference field $\mathbb{C}(x)$, where the only trivial solution of (1.2) is the constant solution $y(x)\equiv-\lambda$.

The purpose of this paper is to prove that his conjecture is true, namely, to prove the

**Theorem 1.** Any nontrivial solution of (1.2) is hypertranscendental over the difference field $\mathbb{C}(x)$.

It should be noted that equation (1.2) may have multiplevalued solutions.

Let $\varphi(x)$ be a nontrivial solution of (1.2). Note that the mapping $\varphi$ has its inverse mapping $\psi$. The mapping $\varphi$ forms the following commutative diagram together with $f(z)=z+1+(\lambda/z)$ and $g(z)=z+1$:

$$
\begin{array}{c}
\varphi \\
\downarrow \\
g
\end{array}
\quad
\begin{array}{c}
f \\
\varphi
\end{array}
$$

so $\psi$ satisfies the commutative diagram

$$
\begin{array}{c}
\psi \\
\downarrow \\
g
\end{array}
\quad
\begin{array}{c}
f \\
\psi
\end{array}
$$

This means that $\psi(x)$ is a solution of the functional equation

$$(1.3) \quad y\left(x+1+\frac{\lambda}{x}\right)=y(x)+1.$$ 

It is clear that equation (1.3) has no constant solution and that the hypertranscendence of $\varphi$ over $\mathbb{C}(x)$ is equivalent to that of $\psi$ over $\mathbb{C}(x)$. Therefore, Theorem 1 is equivalent to the

**Theorem 2.** Any solution of (1.3) is hypertranscendental over the differen-
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tial field $C(x)$.
We shall prepare several preliminary propositions in section 2 and shall carry out the proof of Theorem 2.

§2. Preliminary propositions.
Let us set

$$f(x)=x+1+\frac{\lambda}{x},$$

$\lambda$ being a nonzero constant and denote by $f_n(x)$ the $n$-th iterate of $f(x)$, where $f_0(x)=x$ and $f_1(x)=f(x)$. For any $\alpha \in P$, we define $A(\alpha)$ to be the set

$$\{x \in P; \ f_n(x)=\alpha \text{ for some } n>0\},$$

where $P$ is the extended complex plane. We shall first prove the

Proposition 1. Both $A(0)$ and $A(\infty)$ are infinite pointsets.

Proof. From $f(0)=\infty$ and $f(\infty)=\infty$, it follows that

$$A(\infty)=[0] \cup A(0),$$

hence we have only to show that $A(0)$ is an infinite set.

To do this, it suffices to verify that for any sequence $\{x_1, x_2, \cdots\}$ such that

$$f(x_1)=0, f(x_2)=x_1, \cdots, f(x_k)=x_{k-1}, \cdots,$$

the $x_k$'s are mutually distinct. Suppose that for some positive integers $l>k_x$

$$x_l=x_k.$$

Since

$$f_{l-k}(x_l)=x_k, \text{ we have}$$

$$f_{l-k}(x_k)=x_k,$$

from which follows

$$f_k(f_{l-k}(x_k))=f_k(x_k).$$

From the definition of $\{x_k\}_{k=1}^\infty$, we have

$$f_k(x_k)=0.$$ 

On the other hand,

$$f_k(f_{l-k}(x_k))=f_{l-k}(f_k(x_k))=f_{l-k}(0)=\infty.$$

This is a contradiction. The proposition is thus proved.

Proposition 2. The functional equation

$$a(x+1+\frac{\lambda}{x})=a(x)+c \quad (c: \text{constant and } \lambda \neq 0)$$
has a rational function solution if and only if \( c=0 \). In the case \( c=0 \), any rational function solution is necessarily a constant function.

**Proof.** We shall prove the first assertion. If \( c=0 \), then (2.1) has rational function solutions. In fact \( y(x) \equiv \text{constant} \) is a solution.

Suppose that \( c \neq 0 \) and that (2.1) has a rational function solution \( a(x) \). Taking in (2.1) the limit as \( x \to \infty \), we have

\[
a(\infty) = a(\infty) + c.
\]

This equality implies that \( x=\infty \) is a pole of \( a(x) \). We conclude from this fact that any point \( x \in A(\infty) \) is also a pole of \( a(x) \). By virtue of Proposition 1, \( a(x) \) has infinitely many poles. This contradicts the hypothesis that \( a(x) \) is a rational function.

We next show the second assertion. Suppose that \( a(x) \) is a nonconstant rational function satisfying

\[
a\left(x + 1 + \frac{\lambda}{x}\right) = a(x).
\]  

(2.2)

It is easily verified by the same reason as above that \( x=\infty \) is neither a pole nor a zero of \( a(x) \) and furthermore that \( x=0 \) is neither a pole nor a zero of \( a(x) \). Hence, \( a(x) \) is represented as

\[
a(x) = \frac{c(x)}{d(x)},
\]

where \( c(x) \) and \( d(x) \) are relatively prime polynomials of the same degree \( n>0 \):

\[
c(x) = \sum_{j=0}^{n} c_{j} x^{j}, \quad c_{0}, c_{n} \neq 0,
\]

\[
d(x) = \sum_{j=0}^{n} d_{j} x^{j}, \quad d_{0}, d_{n} \neq 0.
\]

From (2.2) we get

\[
\frac{x^{n} c\left(\frac{x^{2} + x + \lambda}{x}\right)}{x^{n} d\left(\frac{x^{2} + x + \lambda}{x}\right)} = \frac{c(x)}{d(x)}.
\]  

(2.3)

Observing that

\[
x^{n} c\left(\frac{x^{2} + x + \lambda}{x}\right) \quad \text{and} \quad x^{n} d\left(\frac{x^{2} + x + \lambda}{x}\right)
\]

are polynomials of degree \( 2n \) with nonzero constant terms, we see by (2.3) that

\[
x^{n} c\left(\frac{x^{2} + x + \lambda}{x}\right) = 0 \quad \text{and} \quad x^{n} d\left(\frac{x^{2} + x + \lambda}{x}\right) = 0
\]
have \( n \) common roots \( \neq 0 \). This is a contradiction because \( c(x) \) and \( d(x) \) are relatively prime. The proposition is thus completed.

**Proposition 3.** Any rational function solution of the equation

\[
\left( \frac{x^2 - \lambda}{x^2} \right)^h a \left( x+1+\frac{\lambda}{x} \right) = a(x), \quad (\lambda \neq 0)
\]

is identically zero, where \( h \) is a positive integer.

**Proof.** Suppose that \( a(x) \) is a rational function solution which does not identically vanish. We shall show that \( x = \infty \) is neither a pole nor a zero of \( a(x) \). For this purpose, we develop \( a(x) \) into the Laurent expansion at \( x = \infty \):

\[
a(x) = \sum_{j=r}^{\infty} a_j x^{-j}, \quad a_r \neq 0.
\]

Since \( x = \infty \) is a fixed point of

\[
f(x) = x + 1 + \frac{\lambda}{x},
\]

we can obtain the Laurent expansion of

\[
a \left( x+1+\frac{\lambda}{x} \right)
\]

at \( x = \infty \), and then that of

\[
\left( \frac{x^2 - \lambda}{x^2} \right)^h a \left( x+1+\frac{\lambda}{x} \right)
\]

immediately from the expansion of \( a(x) \) as follows

\[
\left( \frac{x^2 - \lambda}{x^2} \right)^h a \left( x+1+\frac{\lambda}{x} \right)
\]

\[
= (1 - \lambda hx^{-2} + \cdots)(a_r x^{-r}(1+x^{-1}+\lambda x^{-2})^{-r} + a_{r+1} x^{-r+1}(1+x^{-1}+\lambda x^{-2})^{-r+1} + \cdots)
\]

\[
= a_r x^{-r} + (a_{r+1} - r a_r) x^{-(r+1)} + \cdots.
\]

Therefore we have

\[
a_{r+1} - r a_r = a_{r+1}, \quad (a_r \neq 0),
\]

from which follows

\[
r = 0.
\]

This shows that \( x = \infty \) is neither a pole nor a zero of \( a(x) \).

Now, by taking the limit in (2.4) as \( x \to 0 \), we have

\[
\left( \frac{x^2}{\lambda} \right)^h a(x) \to a(\infty), \quad (x \to 0).
\]

Hence, \( x = 0 \) is a pole of \( a(x) \) of order \( 2h \).
We shall next show that $a(x)$ has infinitely many poles. Let us choose a sequence of points $x_0, x_1, \cdots, x_k, \cdots, (x_0=0)$ so that
\[
f'(x_1) = x_0, f'(x_2) = x_1, \cdots, f'(x_k) = x_{k-1}, \cdots.
\]
As was shown in the proof of Proposition 1, we have
\[
x_k \neq x'_k, \text{ if } k \neq k'.
\]
Suppose that $x=x_{k-1}$ is a pole of order $l$. Then, $a(x)$ is written as
\[
a(x) = \frac{\partial (x)}{(x-x_{k-1})^l}
\]
in a neighbourhood $U_{k-1}$ of $x=x_{k-1}$. Using (2.5) and (2.4), we shall examine the local behavior of $a(x)$ at $x=x_k$. It is easy to see that
\[
x + 1 + \frac{\lambda}{x} - x_{k-1} = \frac{x_k^{2} - \lambda}{x_k^2} (x-x_k) + \frac{\lambda}{x_k} (x-x_k)^2 + O((x-x_k)^3),
\]
in a neighbourhood $U_k$ of $x=x_k$. Then, from (2.4) and (2.5), we have in $U_k$
\[
a(x) = \left( \frac{x^2 - \lambda}{x^2} \right)^l \left( \frac{x_k^2}{x_k - \lambda} \right) \partial \left( x + 1 + \frac{\lambda}{x} \right) \frac{1}{(x-x_k)^l(1+O(x-x_k))}
\]
if $x_k^2 - \lambda \neq 0$, and
\[
a(x) = \left( \frac{x^2 - \lambda}{x^2} \right)^l \left( \frac{x_k^2}{\lambda} \right) \partial \left( x + 1 + \frac{\lambda}{x} \right) \frac{1}{(x-x_k)^{2l}(1+O(x-x_k))}
\]
if $x_k^2 - \lambda = 0$. Therefore, $x=x_k$ is also a pole of order at least $m = \text{Min}(l, 2l - h)$.

From this fact and the fact that $x=x_0$ is a pole of order $2h$, it follows that $x_k, k=1, 2, \cdots$, are all poles of $a(x)$. Therefore $a(x)$ cannot be a rational function, which proves Proposition 3.

The last proposition is concerned with the following equation
\[
(2.6) \quad \frac{x^2 - \lambda}{x^3} a(x + 1 + \frac{\lambda}{x}) + q(x) = a(x).
\]

**Proposition 4.** Suppose that (i) $q(x)$ is a nonzero rational function (ii) $x=0$ is the only pole of $q(x)$ (iii) the order of the pole $x=0$ is greater than two (iv) $x=\infty$ is a zero point. Then equation (2.6) does not have any rational function solution.

**Proof.** We denote by $m$ the order of pole of $q(x)$ at $x=0$ and by $n$ the order of zero of $q(x)$ at $x=\infty$.

Suppose that (2.6) has a rational function solution $a(x)$. It is easy to see that $a(x)$ is not a constant. In fact, if $a(x) \equiv a$, then
(2.7) \[ \frac{x^2 - \lambda}{x^2} a + q(x) = a. \]

By the assumption on \( q(x) \), the left hand side of (2.7) has a pole of order \( m \geq 3 \) at \( x = 0 \), which is a contradiction.

We see that \( x = 0 \) cannot be a pole of \( a(x) \). In fact, if \( x = 0 \) were a pole of \( a(x) \), then from supposition (ii) we infer in the same way as in the proof of Proposition 3 that \( a(x) \) would have infinitely many poles.

Now, let

(2.8) \[ a(x) = \sum_{j=r}^{\infty} a_j x^{-j}, \quad a_r \neq 0 \]

be the Laurent expansion of \( a(x) \) at \( x = \infty \). Since \( f(0) = \infty \), the Laurent expansion of \( a(x) \) at \( x = 0 \) can be derived from (2.6) and (2.8). Indeed, from

\[
\begin{align*}
  a(x+1+\frac{\lambda}{x}) &= \lambda^{-r} a_r x^r (1+O(x)) \\
  q(x) &= q' x^{-m}(1+O(x)), \quad q' \neq 0
\end{align*}
\]

as \( x \to 0 \), we have

\[
a(x) = \frac{x^2 - \lambda}{x^2} a(x+1+\frac{\lambda}{x}) + q(x) = -\lambda^{-r+1} a_r x^{-r+1}(1+O(x)) + q' x^{-m}(1+O(x)),
\]

as \( x \to 0 \). Since \( x = 0 \) is not a pole of \( a(x) \), we have

\[
  r - 2 = -m \\
  -\lambda^{-r+1} a_r + q' = 0.
\]

From the first relation and \( m \geq 3 \), we get

(2.9) \[ r \leq -1. \]

This means that \( x = \infty \) is a pole of \( a(x) \) of order at least 1.

Finally, we shall consider the Laurent expansion of the left hand side of (2.6) at \( x = \infty \).

Since

\[
\frac{x^2 - \lambda}{x^2} a(x+1+\frac{\lambda}{x}) = a_r x^{-r} + (a_{r+1} - ra_r) x^{-(r+1)} + O(x^{-(r+2)})
\]

and

\[
q(x) = q'' x^{-n}(1+O(x^{-1})), \quad q'' \neq 0
\]

as \( x \to \infty \), we have by (2.9) and \( n \geq 1 \)

\[
\frac{x^2 - \lambda}{x^2} a(x+1+\frac{\lambda}{x}) + q(x) = a_r x^{-r} + (a_{r+1} - ra_r) x^{-(r+1)} + \cdots.
\]
Comparing this expansion with (2.8), we get

\[ a_{r+1} - ra_r = a_{r+1}. \]

This is a contradiction because of \( ra_r \neq 0 \). Thus we have proved Proposition 4.

§ 3. **Proof of Theorem 2.**

The differential ring \( C(x) \{ y \} \) is, by definition, isomorphic to the polynomial ring \( \mathcal{R} = C(x)[y_0, y_1, \ldots] \) in infinitely many usual indeterminates \( y_0, y_1, \ldots \) by the correspondence \( y \rightarrow y_0, y' \rightarrow y_1, \ldots \). Then the set of all algebraic differential equations can be put in a one to one correspondence with \( \mathcal{R} \) in a natural way.

Take an arbitrary solution \( y(x) \) of equation (1.3) and keep it fixed. Suppose that \( y(x) \) satisfies algebraic differential equations. Let \( \mathcal{M} \) be the subset of \( \mathcal{R} \) which corresponds to the set of algebraic differential equations satisfied by \( y(x) \). We want to choose an element of \( \mathcal{M} \) which is simplest in some sense.

For a monomial \( a(x)y_0^{k_0}y_1^{k_1}\cdots y_n^{k_n} \), we call the ordered system \( (k_0, k_1, \ldots, k_n) \) its index and write it simply as

\[ k = (k_0, k'), \quad k' = (k_1, \ldots, k_n). \]

We define \( |k'| \) by

\[ |k'| = k_1 + 2k_2 + \cdots + nk_n. \]

Let us introduce a linear order in the set of indices as follows. Let \( k = (k_0, k') \) and \( l = (l_0, l') \) be indices of monomials \( a(x)y_0^{k_0}y_1^{k_1}\cdots y_n^{k_n} \) and \( b(x)y_0^{l_0}y_1^{l_1}\cdots y_n^{l_n} \). We define that

\[ k > l \]

if either

(i) \( |k'| > |l'| \), or

(ii) \( |k'| = |l'| \) and the first nonzero integer of

\[ k_p - l_p, k_{p-1} - l_{p-1}, \ldots, k_1 - l_1 \]

is positive, where \( p = \text{Max}(n, m) \), or

(iii) \( k' = l' \) and \( k_0 > l_0 \).

In (ii) we made the convention, if \( n < p \), then \( k_{n+1} = \cdots k_p = 0 \) and if \( m < p \), then \( l_{m+1} = \cdots l_p = 0 \). It is easy to see that this order is a linear one. Given a polynomial in \( \mathcal{R} \), we call the greatest index among indices of the nonzero monomials in the polynomial the rank of the polynomial.

We can now choose from \( \mathcal{M} \) a polynomial \( F(x, y_0, y_1, \ldots, y_n) \) of the lowest rank. We write \( F \) as
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\[ F(x, y_0, y_1, \ldots, y_n) \]

\[ = a_{s_0, s'(x)} y_0^{s_0} y_1^{s_1} \cdots y_n^{s_n} + \sum_{k=0}^{s_0-1} a_{k_0, s'(x)} y_0^{k_0} y_1^{s_1} \cdots y_n^{s_n} \]

\[ + \sum_{|k'| = |s'|}^{m(k')} \sum_{k=0}^{k_0} a_{k_0, k'(x)} y_0^{k_0} y_1^{k_1} \cdots y_n^{k_n} \]

where \( a_{s_0, s'(x)} y_0^{s_0} y_1^{s_1} \cdots y_n^{s_n} \) is the monomial of the greatest index. We may assume without loss of generality that

\[ a_{s_0, s'(x)} = 1. \]

It is clear that the polynomial \( F \) is determined uniquely under condition (3.2).

It follows from the definition of \( F \) that

\[ F(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0. \]

Since this relation is an identity in \( x \), we obtain

\[ F\left(x+1+\frac{\lambda}{x}, y\left(x+1+\frac{\lambda}{x}\right), \ldots, y^{(n)}(x+1+\frac{\lambda}{x})\right) = 0. \]

Using equation (1.3), we have

\[ y^{(k)}(x+1+\frac{\lambda}{x}) = \left(\frac{x^2-\lambda}{x^2}\right)^k y^{(k)}(x) + \sum_{l=1}^{k-1} p_{k,l}(x) y^{(l)}(x) \]

where the \( p_{k,l}(x) \)'s are in \( C(x) \), in particular,

\[ p_{k,k-1}(x) = \frac{d}{dx} \left( \sum_{l=1}^{k-1} \left(\frac{x^2}{x^2-\lambda}\right)^l \right). \]

We put

\[ \tilde{F}(x, y_0, y_1, \ldots, y_n) \]

\[ = \left(\frac{x^2-\lambda}{x^2}\right)^{|\lambda|} F\left(x+1+\frac{\lambda}{x}, y_0+1, \frac{x^2}{x^2-\lambda} y_1, \left(\frac{x^2}{x^2-\lambda}\right)^2 y_2 + p_{2,1}(x) y_1, \ldots, \left(\frac{x^2}{x^2-\lambda}\right)^n y_n + \sum_{l=1}^{n-1} p_{n,l}(x) y_l \right). \]

Identities (3.3) and (3.4) show that \( \tilde{F} \) is a polynomial in \( M \) of the same rank as \( F \) and moreover that the coefficient of the monomial of the greatest index in \( \tilde{F} \) is equal to 1. Therefore we have

\[ F(x, y_0, y_1, \ldots, y_n) = \tilde{F}(x, y_0, y_1, \ldots, y_n). \]

Using (3.7), we shall show that we have
This gives a contradiction. Thus the proof of Theorem 2 will be completed.

The rest of this section will be devoted to the proof of (3.8). We first notice that $F$ is of the form

\[
F(x, y_0, y_1, \ldots, y_n) = y_1.
\]

(3.8)

Our proof is divided into five steps.

The first step is to show that $s_0 = 0$. Suppose to the contrary that $s_0 > 0$.

We see by a simple calculation that the coefficient of the term of the index $(s_0, s_1, \ldots, s_n)$ in $F$ is

\[
a_{s_0, s_1, \ldots, s_n} = a_{s_0, s_1, \ldots, s_n} = \frac{x^2 - \lambda}{x^2} p_{s_0, s_1, \ldots, s_n}(x).
\]

(3.10)

From (3.7) we get

\[
a_{s_0, s_1, \ldots, s_n} = a_{s_0-1, s_1, \ldots, s_n} + s_0.
\]

This contradicts Proposition 2. Thus $F$ can be written as

\[
F(x, y_0, y_1, \ldots, y_n) = y_1 \cdots y_n
\]

\[
+ \sum_{k_0=0}^{m(k')} \sum_{k=0}^{n-1} a_{k_0, k'}(x) y_0 y_1 \cdots y_n
\]
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$$+ \sum_{|k'|<|s'|} \sum_{k_0=0}^{m(k')} a_{k_0, k'}(x) y_0 y_1 \cdots y_n.$$ 

The second step is to prove that $m(k')=0$ and $a_{0, k'}(x)=a_{k'}$ (constant) for any $k'$ with $|k'|=|s'|$, $k<s$. Suppose that $m(k')>0$ for some $k'$ with $|k'|=|s'|$, $k<s$. Observing that the coefficients of the terms of the indices $(m(k'), k')$ and $(m(k')-1, k')$ are 

$$a_{m(k'), k'}(x+1+\frac{\lambda}{x})$$
and

$$a_{m(k')-1, k'}(x+1+\frac{\lambda}{x})+m(k') a_{m(k'), k'}(x+1+\frac{\lambda}{x})$$
respectively, we have from (3.7)

$$a_{m(k'), k'}(x+1+\frac{\lambda}{x})=a_{m(k'), k'}(x),$$
$$a_{m(k')-1, k'}(x+1+\frac{\lambda}{x})+m(k') a_{m(k'), k'}(x+1+\frac{\lambda}{x})=a_{m(k')-1, k'}(x).$$

By Proposition 2 and the first equality above, we have

$$a_{m(k'), k'}(x)=a_{m(k'), k'}(x)=\text{constant.}$$

Then, by applying the same proposition to the second equation, we get

$$m(k')=0.$$ 

Hence, $F$ is written as

$$F(x, y_0, y_1, \cdots, y_n)$$

$$=y_1^{s_1} \cdots y_n^{s_n} + \sum_{|k'|<|s'|} a_{k'} y_1 \cdots y_n + \sum_{|k'|<|s'|} \sum_{k_0=0}^{m(k')} a_{k_0, k'}(x) y_0 y_1 \cdots y_n.$$ 

The third step is to see that $m(k')=0$ for any $|k'|<|s'|$. Suppose that $m(k')>0$ for some $k'$ with $|k'|<|s'|$. Let $h'$ be the greatest among such indices. Since the coefficient of the term of the index $(m(h'), h')$ in $\hat{F}$ is

$$(\frac{x^2-\lambda}{x^2})^{s'-|h'|} a_{m(h'), h'}(x+1+\frac{\lambda}{x}),$$

it follows from (3.7) that

$$(\frac{x^2-\lambda}{x^2})^{s'-|h'|} a_{m(h'), h'}(x+1+\frac{\lambda}{x})=a_{m(h'), h'}(x).$$

By Proposition 3, we get

$$a_{m(h'), h'}(x)\equiv 0,$$

which proves the assertion in this step. Thus $F$ is written as
We denote

\[ F(x, y_0, y_1, \ldots, y_n) = y_1^{s_1}\cdots y_n^{s_n} + \sum_{|k| = |s'|} ak y_1^{k_1} \cdots y_n^{k_n} + \sum_{|k| < |s'|} ak'(x) y_1^{k_1} \cdots y_n^{k_n}. \]

The fourth step is to show that \( s_2 = s_3 = \cdots = s_n = 0 \). Suppose that \( s_j > 0 \) for some \( j \geq 2 \). Then the highest monomial of

\[ y_1^{s_1}(y_2+q_{21}y_1)^{s_2}\cdots \left(n + \sum_{l=1}^{n-1} q_{n,l} y_l\right)^{s_n} - y_1^{s_1}y_2^{s_2}\cdots y_n^{s_n} \]

is \( s_r q_{r, r-1} y_1^{s_1} \cdots y_{r-2}^{s_{r-2}} y_{r-1}^{s_{r-1}} y_r^{s_r-1} y_{r+1}^{s_r+1} \cdots y_n^{s_n} \) where \( r = \text{Min} \{ j; s_j > 0, j \geq 2 \} \). We denote by \( T_j, j \geq 2 \) the mapping defined by

\[(k_1, k_2, \cdots) \rightarrow (k_1, \ldots, k_{j-2}, k_{j-1}+1, k_j-1, k_{j+1}, \cdots). \]

We may notice that if both \((*, k')\) and \((*, T_j(k'))\) are indices then

\[ |T_j(k')| = |k'| - 1. \]

We shall find the coefficient of the term of the order \((0, \sigma')\), \( \sigma' = T_r(s'), \) in \( \hat{F} \),

\[ \hat{F} = y_1^{s_1}(y_2+q_{21}y_1)^{s_2}\cdots \left(n + \sum_{l=1}^{n-1} q_{n,l} y_l\right)^{s_n} \]

\[ + \sum_{|k| = |s'|} ak y_1^{k_1} \cdots y_n^{k_n} \]

\[ + \sum_{|k| < |s'|} \left(\frac{x^2 - \lambda}{x^2}\right)^{|s'|-|k'|} y_1^{k_1} \cdots y_n^{k_n}. \]

The coefficients of the terms of that order in the first polynomial and the third one in (3.11) are \( s_r q_{r, r-1} \) and

\[ \frac{x^2 - \lambda}{x^2} a_{\sigma} \left(x + 1 + \frac{\lambda}{x}\right) \]

respectively. The coefficient of the term in the second polynomial in (3.11) is\( \sum_{T_r(s') = T_j(k')} ak' \cdot k_j \cdot q_{j, j-1}(x) \). Since \( r = \text{Min} \{ j; s_j > 0, j \geq 2 \} \) and \( k < s, |k'| = |s'| \), the relation

\[ T_r(s') = T_j(k') \]

is satisfied only for \( r \geq 3, j = 2, k' = T^{-1}_i(\sigma') \). Therefore the coefficient of the term of the order \((0, \sigma')\) in \( \hat{F} \) is

\[ \frac{x^2 - \lambda}{x^2} a_{\sigma} \left(x + 1 + \frac{\lambda}{x}\right) + q_r(x), \]

where

\[ q_r(x) = \begin{cases} s_r q_{r, r-1}(x) + a_{T^{-1}_r(\sigma')} \cdot q_{21}(x), & \text{if } r \geq 3 \\ s_2 q_{21}(x), & \text{if } r = 2. \end{cases} \]
Here, $q_{j,j-1}(x)$ is a rational function defined by (3.5) and (3.10), thus it is written by

$$q_{j,j-1}(x) = -\frac{2\lambda}{x^2} \sum_{i=0}^{j-2} (j-i-1) \left( \frac{x^2 - \lambda}{x^2} \right)^i.$$  

From (3.7) we have

$$\frac{x^2 - \lambda}{x^2} a_\sigma(x + 1 + \frac{\lambda}{x}) + q_r(x) = a_\sigma'(x).$$

By (3.12) and (3.13) it can be easily verified that $q_r(x)$ is a nonzero rational function. Furthermore it has the only pole at $x=0$ of order $2r-1(>2)$ and the zero point at $x=\infty$. Then by virtue of Proposition 4, $a_\sigma(x)$ cannot be a rational function. Therefore $s_2 = s_3 = \cdots = s_n = 0$. Thus $F$ is written as

$$F(x, y_0, y_1, \ldots, y_n) = y_1^{s_1} + \sum_{|k| < s_1} a_k(x) y_1^{k_1} \cdots y_n^{k_n}.$$ 

The final step is to show that $a_k(x) = 0$ for any $|k'| < |s'|$. Let $y_1^{k_1} \cdots y_n^{k_n}$ be the greatest term in $\sum_{|k'| < |s'|} a_{k'}(x) y_1^{k_1} \cdots y_n^{k_n}$. Then the coefficient of $y_1^{k_1} \cdots y_n^{k_n}$ in $F$ is

$$\left( \frac{x^2 - \lambda}{x^2} \right)^{s_1 - |k'|} a_{k'}(x + 1 + \frac{\lambda}{x}).$$

Therefore we obtain from (3.7)

$$\left( \frac{x^2 - \lambda}{x^2} \right)^{s_1 - |k'|} a_{k'}(x + 1 + \frac{\lambda}{x}) = a_{k'}(x).$$

By virtue of Proposition 3, it follows from the relation that

$$a_{k'}(x) \equiv 0,$$

which proves that $F = y_1^{s_1}$.

Since $F$ is, by definition, the lowest polynomial in $M$, $s_1$ is necessarily equal to 1. Thus the proof of Theorem 2 has been completed.

References

[5] T. Kimura; On meromorphic solutions of the difference equation $y(x+1) = y(x) +$
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