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On Liapunov-Razumikhin Type Theorems for Functional Differential Equations

By Junji KATO

(Tohoku University)

1. Introduction.

The aim of this paper is to give complete proofs of the theorems mentioned in [1] and some related remarks.

In the Liapunov direct method for the stability theory of ordinary differential equations, Liapunov functions play central roles. For functional differential equations it is shown by many authors (for example, see [2], [3], [4]) that Liapunov functionals take the place in a natural way. However, the construction of a Liapunov functional is a big problem in practice, and it is not successful even for autonomous linear systems.

Razumikhin [5] (also refer to [2; p. 157]) has established a stability theorem for functional differential equations by utilizing a Liapunov function instead of a Liapunov functional: Consider a system of functional differential equations

$$\dot{x}(t)=F(t, x_t), \tag{1}$$

where

$$x_t(s)=x(t+s), \quad s\in[-h, 0]$$

and $F(t, \phi)$ is continuous on a region in $(-\infty, \infty)\times C([-h, 0], R^n)$.

Theorem A [5]. *Let $F(t, \phi)$ be Lipschitzian in ϕ. Then, a sufficient condition for the zero solution of the system* (1) *to be uniformly asymptotically stable is the existence of a continuous function $v(t, x)$ with the properties*;

(i) *$v(t, x)$ is positive-definite*;

(ii) *$v(t, x)$ admits an infinitely small upper bound*;

(iii) *along each solution $x(t)$ of* (1)

$$D^+v(t, x(t))\leqq -c(|x(t)|) \text{ if } V(t, h; x)\leqq f(v(t, x(t))),$$

where D^+v means the Dini-derivative, $f(u)$ and $c(u)$ are continuous functions with the properties

$$f(u)>u \text{ for } u>0 \text{ and } f(u)-u \text{ is non-decreasing}, \tag{2}$$

$$c(u) \text{ is positive-definite and non-decreasing}, \tag{3}$$

and

$$V(t, \tau; x) = \sup \{v(t+s, x(t+s));\ s \in [-\tau, 0]\}$$

for a constant $\tau \geqq 0$.

The essence of this theorem is the condition (iii). It is instructive to compare the theorem of Razumikhin with theorems of the usual type.

A good reference and related topics can be found in [6].

2. Abstract Liapunov function.

As was seen in Theorem A the conditions for Liapunov functions consist of three parts;

(i) positive-definiteness;

(ii) an infinitely small upper bound;

(iii) a related differential inequality.

The Liapunov function is connected with the system under consideration only through a condition of type (iii), while the role of conditions of type (i) and (ii) is to present an information about the state of the solution from that of a Liapunov function, and vice versa.

In the present note, we shall restrict our consideration to the topics related with a condition of type (iii). Therefore, the argument x contained in the function $v(t, x)$ has a meaning only as a parameter, and there is no difference whether x means a point $x(t) \in R^n$ or a segment $x_t \in C([-h, 0], R^n)$ or even the whole trajectory $x(\cdot)$.

In the following (except in §5 and §7), *a Liapunov function* will mean a continuous, non-negative function $v(t)$, and we shall employ the following notations. Here and henceforth, we set

$$V(t, \tau) = \sup \{v(s);\ t-\tau \leqq s \leqq t\}$$

for a constant $\tau \geqq 0$.

Definition 1. $v(t)$ is said to be *stable*, if for any $\varepsilon > 0$ and any s there exists a $\delta(\varepsilon, s) > 0$ such that

$$v(t) \leqq \varepsilon \quad \text{if} \quad t \geqq s \quad \text{and} \quad V(s, \tau) \leqq \delta(\varepsilon, s).$$

Definition 2. $v(t)$ is *asymptotically stable*, if it is stable and if for any $\varepsilon > 0$, any $\alpha > 0$ and any s there exists a $T(\varepsilon, \alpha, s) \geqq 0$ such that

$$v(t) \leqq \varepsilon \quad \text{if} \quad t-s \geqq T(\varepsilon, \alpha, s) \quad \text{and} \quad V(s, \tau) \leqq \alpha.$$

Definition 3. In the above, if δ and T can be chosen independently of s, the corresponding stability is said to be *uniform*.

By using these definitions, we can reproduce Theorem A in the following way.

Theorem B. *Let $f(u)$ and $c(u)$ be continuous functions with the properties*

(2) and (3), respectively.

Then $v(t)$ is uniformly asymptotically stable, provided that we have

$$D^{+}v(t) \leqq -c(v(t)) \quad \textit{if} \quad V(t, \tau) \leqq f(v(t)).$$

3. Comparison theorems.

Theorem B can be dealt with as a kind of comparison theorem. We shall discuss this more precisely.

Consider an ordinary differential equation

$$\dot{u} = U(t, u) \tag{4}$$

in a scalar u, and denote by $u(t, s, \alpha)$ and $r(t, s, \alpha)$ the maximal solution of (4) through (s, α) to the right and to the left, respectively.

First of all, we shall prove the following lemmas.

Lemma 1. *Suppose that*

$$D^{+}v(t) \leqq U(t, v(t)) \quad \textit{if} \quad v(s) \leqq r(s, t, v(t)) \quad \textit{for all} \quad s \in [t-\tau, t]. \tag{5}$$

Then we have

$$v(t) \leqq u(t, a, \alpha) \quad \textit{for all} \quad t \geqq a$$

whenever $v(s) \leqq r(s, a, \alpha)$ for $s \in [a-\tau, a]$.

Proof. Consider the scalar equation

$$\dot{u} = \begin{cases} U(t, u) & t \leqq a \\ U(t, u) + \varepsilon(t-a) & t > a \end{cases}$$

for an $\varepsilon > 0$, and let $u(t, \varepsilon)$ be a solution of this equation satisfying $u(t, \varepsilon) = r(t, a, \alpha+\varepsilon)$ for $t \leqq a$. Then the standard arguments will give

$$v(t) < u(t, \varepsilon) \quad \text{for} \quad t \geqq a.$$

Thus the proof of Lemma 1 will be completed by letting $\varepsilon \to 0$.

Lemma 2. *Let $f(t, u)$ and $c(t, u)$ be continuous functions which satisfy the conditions (2) and (3) for each fixed t.*

If $v(t)$ satisfies

$$D^{+}v(t) \leqq -c(t, v(t)) \quad \textit{if} \quad V(t, \tau) \leqq f(t, v(t)), \tag{6}$$

then there exists a continuous function $U(t, u)$, $U(t, u) < 0$ for $u > 0$, for which the condition (5) holds. Actually, $U(t, u)$ can be given by

$$U(t, u) = -\min\left\{\frac{1}{3\tau}u, \frac{1}{\tau}\min_{t \leqq s \leqq t+\tau}\left[f\left(s, \frac{2}{3}u\right) - \frac{2}{3}u\right], c(t, u)\right\}. \tag{7}$$

Proof. Since $U(t, u) \geqq -c(t, u)$, it is sufficient to prove that $v(s) \leqq r(s, t, v(t))$ for $s \in [t-\tau, t]$ implies $V(t, \tau) \leqq f(t, v(t))$ or that

$$\sup_{t-\tau\leqq s\leqq t} r(s,t,\alpha)\leqq f(t,\alpha) \quad \text{for any} \quad (t,\alpha), \alpha\geqq 0.$$

Since $U(t,u)\geqq -u/(3\tau)$, we have

$$r(s,t,\alpha)\leqq \alpha e^{-\frac{1}{3\tau}(s-t)}\leqq \alpha e^{\frac{1}{3}}\leqq \frac{3}{2}\alpha \quad \text{for} \quad s\in[t-\tau,t].$$

By the monotonicity, clearly

$$U(t,u)\geqq -\frac{1}{\tau}\min_{t\leqq s\leqq t+\tau}\{f(s,\alpha)-\alpha\}$$

if $u\leqq 3\alpha/2$. Therefore, for $s\in[t-\tau,t]$,

$$\begin{aligned} r(s,t,\alpha)&=\alpha-\int_s^t U(p,r(p,t,\alpha))dp \\ &\leqq \alpha+\frac{1}{\tau}\int_s^t \min_{p\leqq q\leqq p+\tau}\{f(q,\alpha)-\alpha\}dp. \end{aligned}$$

Since $p\in[s,t]\subset[t-\tau,t]$ implies $t\in[p,p+\tau]$, we have

$$r(s,t,\alpha)\leqq \alpha+\frac{1}{\tau}\int_s^t\{f(t,\alpha)-\alpha\}dp\leqq f(t,\alpha)$$

for all $s\in[t-\tau,t]$.

Lemma 3. *If $U(t,u)\leqq 0$ for all (t,u), $u\geqq 0$, if $U(t,u)$ is non-increasing in u for each fixed t and if*

$$(8) \qquad \int^\infty U(t,u)dt=-\infty \quad \textit{for each fixed} \quad u>0,$$

then any non-negative solution of (4) tends to zero as $t\to\infty$.

Proof. Since $U(t,u)\leqq 0$, any non-negative solution $u(t)$ of (4) is non-increasing, and hence there exists a $\beta\geqq 0$ such that $u(t)\to\beta$ as $t\to\infty$ and $u(t)\geqq\beta$ for all t. Therefore, we have

$$\beta-u(a)\leqq\beta-u(t)=\int_t^\infty U(s,u(s))ds\leqq\int_t^\infty U(s,\beta)ds$$

which yields a contradiction unless $\beta=0$.

A sufficient condition for $U(t,u)$ defined by (7) to satisfy the condition (8) is the following: In addition to the conditions (2) and (3) (for each fixed t), the continuous functions $f(t,u)$ and $c(t,u)$ satisfy

$$(9) \qquad f(t,u)-u\geqq k(u)>0 \quad \textit{for} \quad u>0,$$

$$(10) \qquad 0\leqq c(t,u)\leqq K(u),$$

$$(11) \qquad \int^\infty c(t,u)dt=\infty \quad \textit{for each fixed} \quad u>0,$$

where $k(u)$ and $K(u)$ are continuous.

We shall prove this: Put

$$\rho(t, u)=\min\left\{\frac{1}{3\tau}u, \frac{1}{\tau}\min_{t\leqq s\leqq t+\tau}\left[f\left(s, \frac{2}{3}u\right)-\frac{2}{3}u\right]\right\}$$

and

$$\rho(u)=\min\left\{\frac{1}{3\tau}u, k\left(\frac{2}{3}u\right)\right\}\leqq\rho(t, u).$$

Let $\alpha>0$, $s\geqq 0$, $G>0$ be given. Choose $\gamma^*(\alpha, G)$ and $\gamma(s, \alpha, G)$ so that

$$\rho(\alpha)\gamma^*(\alpha, G)\geqq G,$$

$$\int_s^{s+\gamma(s,\alpha,G)} c(t, \alpha)dt\geqq G+|K(\alpha)-\rho(\alpha)|\gamma^*(\alpha, G),$$

and then

$$-\int_s^{s+\gamma(s,\alpha,G)} U(t, \alpha)dt\geqq G.$$

In fact, clearly we have

$$-\int_s^{s+\gamma(s,\alpha,G)} U(t, \alpha)dt=\int_{I_1}\rho(t, \alpha)dt+\int_{I_2}c(t, \alpha)dt,$$

where

$$I_1=\{t\in[s, s+\gamma(s, \alpha, G)];\ \rho(t, \alpha)\leqq c(t, \alpha)\}$$

and $I_2=[s, s+\gamma(s, \alpha, G)]-I_1$. Therefore,

$$\int_{I_1}\rho(t, \alpha)dt\geqq\rho(\alpha)\gamma^*(\alpha, G)\geqq G$$

if $\int_{I_1}dt\geqq\gamma^*(\alpha, G)$, and otherwise

$$-\int_s^{s+\gamma(s,\alpha,G)} U(t, \alpha)dt=\int_s^{s+\gamma(s,\alpha,G)} c(t, \alpha)dt+\int_{I_1}\{\rho(t, \alpha)-c(t, \alpha)\}dt$$

$$\geqq\int_s^{s+\gamma(s,\alpha,G)} c(t, \alpha)dt+\{\rho(\alpha)-K(\alpha)\}\int_{I_1}dt\geqq G.$$

As will be clear from the proof in the above, if the integral in (11) is uniformly divergent, that is, if we can choose $\gamma(\alpha, G)$ for any $\alpha>0$ and $G>0$ so that

$$\int_s^{s+\gamma(\alpha,G)} c(t, \alpha)dt\geqq G$$

for all $s\geqq 0$, then so is the integral in (8).

Now, the following lemma will be proved.

Lemma 4. *In Lemma 3, if the integral in (8) is uniformly divergent, then the approach of the non-negative solutions of (4) to zero is uniform.*

Proof. Let $\alpha>0$ be given, and choose $\gamma(\alpha)$ so that

$$\int_s^{s+\gamma(\alpha)} U\left(t, \frac{\alpha}{2}\right)dt<-\alpha \quad \text{for all} \quad s\geqq 0.$$

Let $u(t)$ be a non-negative solution of (4) satisfying

$$u(s)\leqq\alpha \quad \text{and} \quad u(t)\geqq\frac{\alpha}{2} \quad \text{for} \quad t\in[s, s+\gamma(\alpha)]$$

for a fixed s. Then

$$\begin{aligned} u(s+\gamma(\alpha)) &= u(s)+\int_s^{s+\gamma(\alpha)} U(t, u(t))dt \\ &\leqq \alpha+\int_s^{s+\gamma(\alpha)} U\left(t, \frac{\alpha}{2}\right)dt<0, \end{aligned}$$

which yields a contradiction. Thus $u(s)\leqq\alpha$ implies

$$u(t)\leqq\frac{\alpha}{2} \quad \text{for all} \quad t\geqq s+\gamma(\alpha).$$

Therefore, by choosing an integer $m(\varepsilon, \alpha)$ so that

$$\varepsilon 2^{m(\varepsilon,\alpha)}\geqq\alpha$$

for given $\varepsilon>0$ and $\alpha>0$, we have

$$u(t)\leqq\varepsilon \quad \text{for all} \quad t\geqq s+\sum_{k=0}^{m(\varepsilon,\alpha)}\gamma\left(\frac{\alpha}{2^k}\right),$$

whenever $u(s)\leqq\alpha$.

Thus, we have a generalization of Theorem B.

Theorem 1. *Suppose that the condition (6) holds for continuous functions $f(t,u)$ and $c(t,u)$ which satisfy the conditions (2), (3) for each fixed t and the conditions (9), (10) and (11).*

Then $v(t)$ is asymptotically stable.

Moreover, if the integral in (11) is uniformly divergent, then the asymptotic stability is uniform.

Remark. Clearly, the conditions for $f(t,u)$ and $c(t,u)$ are interchangeable, that is, the conditions (9), (10), (11) in Theorem 1 can be replaced by

(9′) $$c(t,u)\geqq k(u)>0 \quad \text{for} \quad u>0,$$

(10′) $$u\leqq f(t,u)\leqq K(u),$$

(11′) $$\int^{\infty}\{f(t,u)-u\}\,dt=\infty \quad \text{for each fixed} \quad u>0.$$

Moreover, under the condition (9) $f(t,u)$ can be replaced by a function which is independent of t, and the same is true for $c(t,u)$ under the condition (9′).

4. The case where $c=0$.

Lemma 2 is valid even if $c(t,u)=0$ or $f(t,u)=u$. But in this case the relation (7) gives $U(t,u)=0$, and Lemma 1 shows that

$$v(t)\leqq V(a,\tau) \quad \text{for all} \quad t\geqq a,$$

that is, $V(t,\tau)$ is non-increasing in t.

In this section, we shall present further results in the case $c(t,u)=0$.

Theorem 2. *Let $f(u)$ be a continuous, increasing function such that $f(u)>u$ for $u>0$ and $f(0)=0$.*

Suppose that

$$D^+v(t)\leqq 0 \quad \textit{if} \quad V(t,\tau)\leqq f(v(t)).$$

Then we have

(i) $v(t)\leqq\max\{v(s),f^{-1}(V(s,\tau))\}$ *for all* $t\geqq s$,

(ii) $\lim_{t\to\infty} v(t)$ *exists.*

Proof. (i) Suppose that the conclusion is not true. Then there exists a $t_1>s$ with

$$v(t_1)>\max\{v(s),f^{-1}(V(s,\tau))\},$$

and hence we can find an $r\in[s,t_1]$ such that

$$D^+v(r)>0 \quad \text{and} \quad v(r)\geqq f^{-1}(V(s,\tau)).$$

Since $V(r,\tau)\leqq V(s,\tau)$ as mentioned in the above, we have $f(v(r))\geqq V(r,\tau)$, which yields a contradiction.

(ii) Since $0\leqq v(t)\leqq V(a,\tau)$ for all $t\geqq a$,

$$0\leqq\beta(=\varliminf_{t\to\infty} v(t))\leqq\alpha(=\varlimsup_{t\to\infty} v(t))<\infty.$$

Suppose that $\alpha>\beta$. For every $\varepsilon>0$ we can find a $T(\varepsilon)$ such that

$$V(t,\tau)\leqq\alpha+\varepsilon \quad \text{for all} \quad t\geqq T(\varepsilon),$$

because $\varlimsup_{t\to\infty} V(t,\tau)=\varlimsup_{t\to\infty} v(t)=\alpha$ for a finite $\tau\geqq 0$. Let $\varepsilon>0$ be chosen so small that $f(\alpha-\varepsilon)>\alpha+\varepsilon$ and that $\alpha>\beta+\varepsilon$. Since it is possible to find an $s\geqq T(\varepsilon)$ such that $v(s)<\alpha-\varepsilon$, the property (i) implies

$$v(t)\leqq\max\{v(s),f^{-1}(V(s,\tau))\}<\alpha-\varepsilon \quad \text{for all} \quad t\geqq s,$$

which contradicts the fact that $\varlimsup_{t\to\infty} v(t)=\alpha$.

Thus we must have $\alpha=\beta$.

Theorem 3. *In addition to the conditions in Theorem 2, assume that for any sequence* $\{t_m\}, t_m\to\infty$, $D^+v(t_m)$ *does not converge to zero if* $v(s+t_m)$ *converges to a non-zero constant for* $s\in[-\tau, 0]$ *and* $f(v(t_m))\geqq V(t_m, \tau)$.

Then $v(t)$ *is uniformly asymptotically stable.*

Proof. The uniform stability of $v(t)$ is clear. Now, we shall show that for any $\alpha>0$ there exists a $T(\alpha)\geqq 0$ for which

$$f(v(t))\leqq V(s, \tau) \quad \text{for all} \quad t-s\geqq T(\alpha) \tag{12}$$

if $f^{-1}(\alpha)\leqq V(s, \tau)\leqq\alpha$ and $s\geqq 0$.

Suppose this is false. Then there exists a sequence $\{s_m\}$ such that

$$f^{-1}(\alpha)\leqq V(s_m, \tau)\leqq\alpha, \quad s_m\geqq 0$$

and that

$$f(v(t))>V(s_m, \tau) \quad \text{for a} \quad t\geqq s_m+2m.$$

The last inequality implies

$$f(v(s_m+t))>V(s_m, \tau) \quad \text{for all} \quad t\in[0, 2m],$$

because if $f(v(t^*))<V(s_m, \tau)$ for a $t^*\geqq s_m$, we have $f(v(t))\leqq\max\{f(v(t^*)), V(t^*, \tau)\}\leqq V(s_m, \tau)$ for all $t\geqq t^*$ by (i) of Theorem 2. Therefore

$$f(v(s_m+t))>V(s_m+t, \tau) \quad \text{for all} \quad t\in[0, 2m],$$

because $V(t, \tau)$ is non-increasing. Hence

$$D^+v(s_m+t)\leqq 0,$$

that is, $v(s_m+t)$ is non-increasing on $[0, 2m]$. Since

$$f^{-2}(\alpha)-\alpha\leqq v(s_m+2m)-v(s_m+m)\leqq 0,$$

we can find a $t_m\in[s_m+m, s_m+2m]$ such that

$$v(t_m)-v(t_m-\tau)\to 0, D^+v(t_m)\to 0 \quad \text{as} \quad m\to\infty,$$

where $f^{-m}(\alpha)=f^{-1}(f^{-(m-1)}(\alpha))$. On the other hand, since $\alpha\geqq v(t_m)\geqq f^{-2}(\alpha)$, we can select a subsequence $\{v(t_{m_k})\}$ which converges to a non-zero constant c. Thus, we have

$$f(v(t_{m_k}))\geqq V(t_{m_k}, \tau), t_{m_k}\to\infty, D^+v(t_{m_k})\to 0, v(t_{m_k}+s)\to c$$

as $k\to\infty$ for any $s\in[-\tau, 0]$, which contradicts the assumption.

It is obvious that $f^{-m}(\alpha)$ tends to zero as $m\to\infty$ for any fixed $\alpha>0$, and hence there exists an integer $m(\alpha, \varepsilon)$ for a given $\alpha>0$ and $\varepsilon>0$ such that $f^{-m}(\alpha)\leqq\varepsilon$ for all $m\geqq m(\alpha, \varepsilon)+1$.

Now we can define a desired number $T(\varepsilon, \alpha)$ by

$$T(\varepsilon, \alpha) = T(\alpha) + T(f^{-1}(\alpha)) + \cdots + T(f^{-m(\alpha,\varepsilon)}(\alpha)) + m(\alpha, \varepsilon)\tau$$

for the number $T(\alpha)$ in (12). In fact, if $f^{-k-1}(\alpha) \leqq V(s, \tau) \leqq f^{-k}(\alpha)$, we have

$$V(s + T(f^{-k}(\alpha)) + \tau, \tau) \leqq f^{-k-1}(\alpha),$$

and hence $V(a, \tau) \leqq \alpha$ implies

$$v(t) \leqq V(t, \tau) \leqq f^{-m(\alpha,\varepsilon)-1}(\alpha) \leqq \varepsilon \quad \text{for all} \quad t \geqq a + T(\varepsilon, \alpha).$$

Remark. When $v(t) = v(t, x)$ depends on a parameter x, to get the uniformity of asymptotic stability with respect to x, the assumption in Theorem 3 should be modified in the following way: For any sequence $\{t_m, x_m\}$, $t_m \to \infty$, $D^+v(t_m, x_m)$ never converges to zero if $v(t_m + s, x_m)$ converges to a non-zero constant and if $f(v(t_m, x_m)) \geqq V(t_m, \tau; x_m) = \sup\{v(t_m + s, x_m); s \in [-\tau, 0]\}$.

Yorke [7] has given a theorem of the same type.

5. The case where $f = u$.

When $v(t)$ satisfies

$$(13) \qquad D^+v(t) \leqq -c^*(t, v(t)) \quad \text{if} \quad v(t) \geqq V(t, \tau)$$

for a continuous function $c^*(t, u)$ with the property (3) for each fixed t, we may suspect the existence of continuous functions $f(t, u)$ and $c(t, u)$ as those in Lemma 2.

However, to show this, we must investigate how $v(t)$ depends on the parameters.

Suppose that for a function $x(\cdot)$ under consideration (a solution of the system (1) or not) $v(t) = v(t, x)$ depends on the h_1-segment of $x(\cdot)$; $x(t+s)$ for $s \in [-h_1, 0]$ and $D^+v(t, x)$ on the h_2-segment of $x(\cdot)$, where $h_2 \geqq h_1 \geqq 0$. Since the relation $v(t, x) \geqq V(t, \tau; x)$ is affected by the $(h_1 + \tau)$-segment of $x(\cdot)$ only, τ will be chosen so that $h_1 + \tau \geqq h_2$.

Assume the following condition: There exist two functions $\eta(t, \varepsilon)$ and $f^*(t, u, \varepsilon)$ for any $\varepsilon > 0$ such that $f^*(t, u, \varepsilon)$ satisfies the condition (2) for fixed (t, ε) and that

$$(14) \quad |D^+v(t, x) - D^+v(t, y)| \leqq \varepsilon \quad \text{if} \quad \sup_{-h_2 \leqq s \leqq 0} |x(t+s) - y(t+s)| \leqq \eta(t, \varepsilon),$$

(15) for any (t, x, ε) with $f^*(t, v(t, x), \varepsilon) \geqq V(t, \tau; x)$ there exists a y for which

$$v(t, y) = V(t, \tau; y) \geqq v(t, x) \quad \text{and} \quad \sup_{-h_2 \leqq s \leqq 0} |x(t+s) - y(t+s)| \leqq \varepsilon.$$

Here clearly $\eta(t, \varepsilon)$ and $f^*(t, u, \varepsilon)$ can be assumed to be non-decreasing in ε for fixed (t, u). Hence the function $f(t, u)$ defined by

$$f(t, u)=f^*\left(t, u, \eta\left(t, \frac{1}{2}c^*(t, u)\right)\right)$$

satisfies the condition (2) for each fixed t. If

$$f(t, v(t, x))\geqq V(t, \tau; x),$$

we can find a y such that $v(t, y)=V(t, \tau; y)\geqq v(t, x)$ and that

$$\sup_{-h_1\leqq s\leqq 0}|x(t+s)-y(t+s)|\leqq\eta\left(t, \frac{1}{2}c^*(t, v(t, x))\right).$$

Therefore (13) and (14) imply

$$D^+v(t, x)\leqq -c^*(t, v(t, y))+\frac{1}{2}c^*(t, v(t, x))\leqq -\frac{1}{2}c^*(t, v(t, x)).$$

Thus, we have the relation (6) with $c(t, u)=c^*(t, u)/2$.

Theorem 4. *Suppose that $v(t, x)$ satisfies the conditions* (13), (14) *and* (15).

Then, there exist continuous functions $f(t, u)$ and $c(t, u)$ with all properties given in Lemma 2.

Remark. In the case where $h_1=0$, the condition (15) will be satisfied if $\partial v/\partial x\neq 0$ at every point (t, x) with $v(t, x)>0$.

A theorem of the above type was shown in [5], but in [5] the condition (15) is not mentioned explicitly.

6. A relationship between the theorems of Liapunov type and of Razumikhin type.

In this section, we shall show that an additional condition on $f(t, u)$ is sufficient to construct a Liapunov function which satisfies a differential inequality of the usual type.

Theorem 5. *Let $f(t, u)$ and $c(t, u)$ be continuous functions such that $f(t, u)>u$ and $c(t, u)>0$ for $u>0$ and that $f(t, u)/u$ and $c(t, u)$ are non-decreasing for fixed t.*

If a Liapunov function $v(t)$ satisfies the condition (6), *then there exists a Liapunov function $w(t)$ satisfying*

$$v(t)\leqq w(t)\leqq V(t, \tau) \tag{16}$$

and

$$D^+w(t)\leqq -c^*(t, w(t)) \tag{17}$$

for a continuous function $c^*(t, u)$ with the property (3) *for each fixed t.*

Proof. Putting

$$\alpha(t, u)=\frac{1}{\tau}\log\left[\inf_{-\tau\leqq s\leqq 0}\frac{u}{f_u^{-1}(t-s, u)}\right],$$

it will be shown that the function $w(t)$ defined by

$$w(t)=\sup_{-\tau\leqq s\leqq 0} e^{\alpha(t+s, v(t+s))s}v(t+s)$$

has the required properties, where $f_u^{-1}(t, u)$ denotes the inverse function of $f(t, u)$ with respect to u. Here we should note that $f(t, u)$ is increasing in u and hence $f_u^{-1}(t, u)$ exists.

Since $\alpha(t, u)\geqq 0$, the relation (16) is immediate.

We shall discuss the inequality (17). Clearly we can choose an $s(t, h)\in[-\tau, 0]$ for any $h>0$ so that

$$w(t+h)=e^{\alpha(\rho(t,h), v(\rho(t,h)))s(t,h)}v(\rho(t, h)),$$

where $\rho(t, h)=t+h+s(t, h)$ for brevity. Furthermore, $s(t, h)$ can be assumed to tend $s(t, 0)$ as $h\to 0$.

Case 1. $s(t, h)+h\leqq 0$ *for every small* $h>0$.
Under the assumption,

$$w(t)\geqq e^{\alpha(\rho(t,h), v(\rho(t,h)))(h+s(t,h))}v(\rho(t, h)),$$

and hence

$$w(t+h)-w(t)\leqq w(t+h)\{1-e^{\alpha(\rho(t,h), v(\rho(t,h)))h}\}.$$

Since $u/f_u^{-1}(t, u)$ is non-decreasing in u, $\alpha(t, u)$ is non-decreasing in u, too. Therefore, by noting $w(t+h)\leqq v(\rho(t, h))$, we have

$$w(t+h)-w(t)\leqq w(t+h)\{1-e^{\alpha(\rho(t,h), w(t+h))h}\},$$

which implies that

$$D^+w(t)\leqq-\alpha(\rho(t, 0), w(t))w(t).$$

Case 2. $s(t, h)+h>0$ *for an* h *arbitrary small.*
In this case, clearly $s(t, 0)=0$, that is, $w(t)=v(t)$. Moreover, for any $s\in[-\tau, 0]$

$$v(t)\geqq e^{\alpha(t+s, v(t+s))s}v(t+s)\geqq e^{-\alpha(t+s, v(t+s))\tau}v(t+s)$$

$$\geqq\frac{f_u^{-1}(t, v(t+s))}{v(t+s)}\cdot v(t+s)=f_u^{-1}(t, v(t+s)),$$

that is, $f(t, v(t))\geqq V(t, \tau)$. Therefore, by the assumption

$$D^+v(t)\leqq-c(t, v(t))=-c(t, w(t)).$$

On the other hand, since

$$w(t+h)-w(t)=w(t+h)-v(\rho(t, h))+v(t+h+s(t, h))-v(t)$$

$$= \{e^{\alpha(\rho(t,h),v(\rho(t,h)))s(t,h)}-1\}v(\rho(t,h)) \\ +v(t+h+s(t,h))-v(t),$$

we have

$$\frac{w(t+h)-w(t)}{h} \leqq \alpha(\rho(t,h), v(\rho(t,h)))v(\rho(t,h))\frac{s(t,h)}{h} \\ +D^+v(t)\left\{1+\frac{s(t,h)}{h}\right\}+\varepsilon(t,h) \\ \leqq \max\{-\alpha(\rho(t,h), v(\rho(t,h)))v(\rho(t,h)), D^+v(t)\}+\varepsilon(t,h),$$

because

$$-1 \leqq \frac{s(t,h)}{h} \leqq 0 \quad \text{and} \quad A\lambda+B(1-\lambda)\leqq \max\{A, B\} \quad \text{for} \quad \lambda\in[0,1],$$

where $\varepsilon(t,h)\to 0$ as $h\to 0$. From this it follows

$$D^+w(t)\leqq -\min\{\alpha(t, w(t))w(t), c(t, w(t))\}.$$

Thus, by putting

$$c^*(t,u)=\min\{c(t,u), \inf_{-\tau\leqq s\leqq 0} \alpha(t+s, u)u\}$$

the condition (17) holds good.

7. Stability problem of a system.

Now back to the stability problem of the zero solution of the system (1).

Consider a Liapunov function $v(t,x)$ which depends on the solution $x(\cdot)$ of the system (1). As was stated in Section 2, under the assumption that $v(t,x)$ is positive definite, that is,

$$v(t,x)\geqq a(|x(t)|) \tag{18}$$

for a positive definite function $a(\cdot)$, $x(t)$ keeps small or tends to zero as $t\to\infty$ according to the same behavior of $v(t,x)$.

Therefore, under the condition (18), in order to show the (asymptotic) stability of the zero solution of (1) from that of $v(t,x)$, we must give a condition to guarantee

$$V(a+\tau, \tau; x)\to 0 \quad \text{as} \quad \sup_{-h\leqq s\leqq 0}|x(a+s)|\to 0.$$

If $v(t,x)$ admits an infinitely small upper bound, that is, $v(t,x)\leqq b(\sup_{-h\leqq s\leqq 0}|x(t+s)|)$ for a continuous function $b(\cdot)$, $b(0)=0$, this follows from the fact

$$\sup_{-\tau-h\leqq s\leqq 0}|x(a+\tau+s)|\to 0 \quad \text{as} \quad \sup_{-h\leqq s\leqq 0}|x(a+s)|\to 0. \tag{19}$$

It is obvious that (19) holds if and only if the zero solution of the system

(1) is unique for the initial value problem.

Furthermore, in order that (19) holds uniformly in the choice of a, a necessary and sufficient condition is the uniqueness of the zero solution of every system

$$\dot{x}(t)=G(t, x_t),\quad G(t, \phi)\in H(F),$$

for the initial value problem, whenever the hull $H(F)$ of $F(t, \phi)$ is compact in the compact-open topology (see [8; Lemma 1], and for a related problem see [9]), where $G(t, \phi)\in H(F)$ means that for a sequence $\{s_k\}$, $F(t+s_k, \phi)$ converges to $G(t, \phi)$ uniformly on any compact set in the domain of $F(t, \phi)$. The latter condition holds if $F(t, \phi)$ satisfies a Lipschitz condition or

$$|F(t, \phi)|\leqq L \sup_{-h\leqq s\leqq 0} |\phi(s)| \tag{20}$$

for a constant $L>0$.

Combining the above, we have the following:

Theorem 6. *Let $v(t, x)$ be a Liapunov function with the properties;*

(i) *positive-definite,*

(ii) *admits an infinitely small upper bound,*

and suppose that the zero solution of the system (1) *is unique for the initial value problem.*

Then the zero solution of (1) *is (asymptotically) stable if $v(t, x)$ is (asymptotically) stable.*

Moreover, if $F(t, \phi)$ satisfies the condition (20), *then the uniformity of the stability of $v(t, x)$ implies that of the zero solution of* (1).

Remark. In the above, we understand that the stability of $v(t, x)$ is uniform in the parameter x. Clearly this fact is satisfied if $f(t, u)$ and $c(t, u)$ is independent of x in Theorems 1 and 2 (also, refer the remark of Theorem 3).

8. Instability.

If it is possible to show that $v(t)$ satisfies

$$V(t, \tau)\leqq f(t, v(t)) \quad \text{for all } t, \tag{21}$$

then the relation (6) becomes a differential inequality of the usual type

$$D^+v(t)\leqq -c(t, v(t)).$$

In the case where $v(t)=v(t, x(t))$ for a solution $x(\cdot)$ of (1), the inequality (21) means that the set

$$\Omega=\{\phi\in C([-\tau, 0], R^n);\ \sup_{-\tau\leqq s\leqq 0} v(t+s, \phi(s))\leqq f(t, v(t, \phi(0)))\}$$

is invariant along the solutions of (1).

Actually, the proof of Theorem A (see [2]) has been carried out by showing that the solution $x(t)$ never goes away from the set Ω as long as

$$v(t, x(t))\geqq \gamma$$

for a $\gamma>0$.

For the stability theorem, it is sufficient to prove that the boundary of a neighborhood of the solution under consideration is never reached by a solution starting from inside. However, for the instability theorem, we must find a solution which reaches the boundary of a given neighborhood. Therefore, for a condition of the type;

$$D^+v(t, x)\geqq c(t, v(t, x)) \quad \text{as long as} \quad x\in\Omega,$$

we must show that Ω is invariant as well as the fact that $\partial\Omega$ contains the zero.

In terms of the abstract Liapunov function, we have the following obvious theorem, where $v(t)$ is said to be *unstable* if $v(t)$ becomes arbitrarily large. It should be noted that for $v(t)=v(t, x(t))$ with the solution $x(\cdot)$ of (1), the instability of $v(t)$ implies that of the zero solution of (1), if

$$v(t, x)\leqq b(|x|) \quad \text{and} \quad 0\in\partial\Omega.$$

Theorem 7. *Suppose that for the Liapunov function $v(t)$ we have*

$$D^+v(t)\geqq c(t, v(t)) \quad \textit{as long as} \quad v(t)\in P_t \tag{22}$$

for a property P_t, where $c(t, u)$ is of the type (3) in u and (11) holds.

If the property P_t is invariant, that is, $v(t_0)\in P_{t_0}$ implies $v(t)\in P_t$ for all $t\geqq t_0$ and if $v(t_0)\in P_{t_0}$ holds with $v(t_0)>0$, then $v(t)$ is unstable.

Theorem 7 is generalization of the theorem given by Hale [3, Theorem 4], in which $v(t)\in P_t$ means $v(t)>0$. In the case, the fact that P_t is invariant is clear under the condition (22).

It is of Razumikhin type, if $v(t)\in P_t$ means

$$f(v(t))\geqq V(t, \tau)$$

for a non-decreasing function $f(u)\geqq u$. Moreover, we have the following, which is a generalization of the idea seen in [10, Theorem 8].

Theorem 8. *Suppose that two Liapunov functions $v(t)$ and $w(t)$ satisfy*

$$D^+v(t)\geqq c(t, v(t)) \quad \textit{and} \quad D^+w(t)<c(t, v(t))$$

as long as

$$f(v(t))\geqq \max\{V(t, \tau), W(t, \tau)\}. \tag{23}$$

If the relation (23) with $v(t_0)>0$ holds at $t=t_0$, then $v(t)$ is unstable.

Here

$$W(t, \tau)=\sup\{w(s);\ s\in[t-\tau, t]\}.$$

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