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On a Certain Hypergeometric Differential System (II)

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1. Inrtoduction.

This paper is devoted to the study to the partial differential system associated with Lauricella's function $F_D^{(n)}$ [9], (where n denotes the number of variables) further to that discussed by the author in [7] where, althogh certain of the solutions of the system were given, others, indispensible to its complete integration, were overlooked.

The function $F_D^{(n)}$ is of particular interest from the point of view of, for example, the study of certain elliptical and hyperelliptical integrals [1], the Euler-Poisson equation in several variables [2] and [3], the expectation of an arbitrary power of any given quadratic form in a normal population of several variables [4] and the study of the Lie algebra $SL(n+3C)$ [10] which is connected with certain aspects of atomic and elementary particle physics.

For brevity in the prelimaries, the reader is referred to [7]. The case $n=2$ has been exhaustively investigated by Erdélyi [6]. Throughout this paper, it is assumed that the parameters and variables are all such that any series involved are convergent, and that exceptional values of the parameters which make any of the gamma-functions infinite are tacitly excluded.

2. The case where $n=3$.

All the solutions of the system under consideration when $n=3$ may be expressed as

$$\int_C u^{\beta_1+\beta_2+\beta_3-\gamma}(u-1)^{\gamma-\alpha-1}(u-x)^{-\beta_1}(u-y)^{-\beta_2}(u-z)^{-\beta_3}du \tag{2.1}$$

(cf. [7] eq. (4.1))

where C is a Pochhammer double-loop with at least one of the six singularities of the integrand inside one loop.

This gives three possibilities for the form of C:

(i) $[a; b]$

(ii) $[a; b, c]$

(iii) $[a, b; c, d]$

where a, b, c, d, e, f represents any permutation of the singularities of the integrand:

$$0, 1, \infty, x, y, z.$$

The singularities not mentioned in each case are taken to be outside the contour; $[a_1, a_2, \cdots; b_1, b_2, \cdots]$ is a short-hand form of $a_1, a_2, \cdots$ in one loop, and $b_1, b_2, \cdots$ in the other loop of the contour of the type under consideration.

Cases (i) and (ii) have already been dealt with:

(i) yields $\binom{6}{2}=15$ different possibilities, each of the form $F_D^{(n)}$.

(ii) yields $6\binom{5}{2}=60$ different possibilities, each of the form G_B [11], or its equivalent $C_n^{(1)}$ or $C_n^{(2)}$ [7].

If we consider the number of singularities in each loop and outside the contour respectively as P, Q, R, then, in each of the previous cases, at least one of these three numbers is unity. For $n \geqq 3$, it is clearly possible that valid contours exist for which all of P, Q, R exceed unity, and it is these solutions which were overlooked by the author in [7].

The case (iii), considered now in some slight detail, gives rise to $1/3\binom{6}{2}\binom{4}{2}=30$ distinct solutions.

We consider the integral

$$(2.2)\qquad \int_{[a,b;c,d]} u^{\beta_1+\beta_2+\beta_3-\gamma}(u-1)^{\gamma-\alpha-1}(u-x)^{-\beta_1}(u-y)^{-\beta_2}(u-z)^{-\beta_3}du$$

and by means of a simple bi-linear transformation of the variable of integration (cf. [6]), (2.2) without loss of generality reduces to

$$(2.3)\qquad R(x, y, z)\int_{[0,s';1,s]} u^{-\lambda}(u-1)^{-\mu}(u-1-s)^{-\nu}(u-s')^{-\nu'}(u-s'')^{-\nu''}du$$

where $R(x,y,z)=x^{\sigma}(1-x)^{\rho}y^{\sigma'}(1-y)^{\rho'}z^{\sigma''}(1-z)^{\rho''}(x-y)^{\tau}(x-z)^{\tau}(y-z)^{\tau}$ and λ, μ, ν etc. depend upon the parameters $\alpha, \beta_1, \beta_2, \beta_3, \gamma$ and s, s', and s'' are rational functions of x, y, z. The factors of the integrand of (2.3) are now expanded by means of the formulae

$$(2.4)\qquad \begin{cases} (u-1-s)^{-\nu}=(u-1)\displaystyle\sum_{m=0}^{\infty}\frac{(\nu)_m}{m!}\left(\frac{s}{u-1}\right)^m, \\ (u-s')^{-\nu'}=u^{-\nu'}\displaystyle\sum_{n=0}^{\infty}\frac{(\nu')_n}{n!}\left(\frac{s'}{u}\right)^n, \\ (u-s'')^{-\nu''}=\displaystyle\sum_{p=0}^{\infty}\frac{(\nu'')_p}{\rho!}(us'')^p \end{cases}$$

which series converge uniformly on the contour, after it has been suitably deformed, if s, s', and s'' are each of modulus less than unity. Term-by-term integration is thus allowable and the integral in (2.3) takes the form

$$\text{(2.5)} \qquad \sum_{m,n,p=0}^{\infty} \frac{(\nu)_m(\nu')_n(\nu'')_p}{m!\,n!\,p!} s^m s'^n s''^p (-1)^{n+p+\lambda+\nu'} \times \int_{[0;1]} (-u)^{\lambda-\nu'-n+p}(u-1)^{-\mu-\nu-m}du$$

The integral of (2.5) may be evaluated by means of the formula [5]

$$\text{(2.6)} \qquad \int_{[0;1]} (-u)^{\alpha-1}(u-1)^{\beta-1}du = \frac{(2\pi i)^2}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha+\beta)}$$

and so may be written

$$\frac{(2\pi i)^2}{\Gamma(\lambda+\nu')\Gamma(\mu+\nu)\Gamma(2-\lambda-\mu-\nu-\nu')} \cdot \frac{(\lambda+\mu+\nu+\nu'-1)_{m+n-p}}{(\lambda+\nu')_{n-p}(\mu+\nu)_m}$$

Hence, the typical solution under consideration becomes (apart from a constant factor)

$$\text{(2.7)} \qquad R(x, y, z) D_{(3)}^{1,3}(\lambda+\mu+\nu+\nu'-1, \nu'', \nu', \nu;\ \lambda+\nu', \mu+\nu;\ s'', s', -s),$$

where

$$D_{(3)}^{1,3}(a, b_1, b_2, b_3;\ c, c';\ x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{n+p-m}(b_1)_m(b_2)_n(b_3)_p}{(c)_{p-m}(c')_n} \frac{x^m y^n z^p}{m!\,n!\,p!}.$$

It seems worthwhile observing that this type of function was encountered by the author while studying the analytical continuation of $F_D^{(3)}$ near the point $(0, 1, \infty)$, which ultimately leads to the work of this paper.

3. The case $n=4$.

All the relevent solutions in this case may be written as

$$\text{(3.1)} \qquad \int_C u^{\beta_1+\beta_2+\beta_3-\gamma}(u-1)^{\gamma-\alpha-1}(u-x)^{-\beta_1}(u-y)^{-\beta_2}(u-z)^{-\beta_3}(u-t)^{-\beta_4}du$$

where C is the same type of contour as in the last section, and it may take the following four forms:

(i) $[a; b]$
(ii) $[a; b, c]$
(iii) $[a; b, c, d]$
(iv) $[a, b; c, d]$

where a, b, c, d, e, f, g represent any permutation of the seven singularities of the integrand

$$0, 1, \infty, x, y, z, t \qquad \text{(cf. [7] section 4).}$$

(i) yields $\binom{7}{2}=21$ solutions of type $F_D^{(4)}$.

(ii) yields $7\binom{6}{2}=105$ solutions of type D_4 [7],

(iii) yields $7/2\binom{6}{3}=70$ solutions of type D_5 [7].

Case (iv) gives rise to $1/2\binom{7}{2}\binom{5}{2}=105$ solutions of a new type; a typical example is

$$(3.2)^{\cdot} \quad \int_{[0,x;1,y]} u^{\beta_1+\beta_2+\beta_3+\beta_4-\gamma}(u-1)^{\gamma-\alpha-1}(u-x)^{-\beta_1}(u-y)^{-\beta_2}(u-z)^{-\beta_3}(u-t)^{-\beta_4}du$$

which is reducible, without loss of generality, to the form (cf. (2.3)).

$$(3.3) \qquad R(x, y, z, t)\int_{[0,s';1,s]} u^{-\lambda}(u-1)^{-\mu}(u-1-s)^{-\nu}(u-s')^{-\nu'} \times(1-us'')^{-\nu''}(1-us''')^{-\nu'''}du$$

and as in the previous section, if the moduli of s, s', s'', and s''' are each less than unity, the integral of (3.3) may be written

$$(3.4) \qquad \sum_{l,m,n,p}^{\infty} \frac{(\nu)_l(\nu')_m(\nu'')_n(\nu''')_p}{l!m!n!p!} s^l s'^m s''^n s'''^p(-1)^{m+n+p+\lambda+\nu'} \times\int_{[0;1]}(-u)^{-\lambda-\nu'-m+n+p}(u-1)^{-\mu-\nu-l}du$$

By the application of (2.6), the corresponding solution is seen to be (apart from a constant factor)

$$(3.5) \quad R(x, y, z, t)D_{(4)}^{2,3}(\lambda+\mu+\nu+\nu', -1, \nu, \nu'\nu'', \nu'''; \lambda+\nu', \mu+\nu; s''', s'', s', -s),$$

where

$$(3.6) \qquad D_{(4)}^{2,3}(a, b_1, b_2, b_3, b_4; c, c'; x, y, z, t) = \sum_{l,m,n,p=0}^{\infty} \frac{(a)_{n+p-l-m}(b_1)_l(b_2)_m(b_3)_n(b_4)_p}{(c)_{n-l-m}(c')_p} \frac{x^l y^m z^n t^p}{l!\,m!\,n!\,p!}$$

Each of the solutions given in sections 2 and 3 has several different expansions which are not considered worthwhile of explicit mention here.

4. The case for general n.

The solution may be expressed as

$$(4.1) \qquad J=\int_C u^{\beta_1+\cdots+\beta_n-\gamma}(u-1)^{\gamma-\alpha-1}(u-x_1)^{-\beta_1}\cdots(u-x_n)^{-\beta_n}du.$$

Let the variables $x_1, \cdots, x_n$ be partitioned into three sets

(a) $x_1, \cdots, x_p$

(b) $x_{p+1}, \cdots, x_q$

(c) $x_{q+1}, \cdots, x_n$,

at each of which, together with the points $0, 1, \infty$ we have a singularity of the integrand, so that, without loss of generality, the contour of integration may be taken to be a Pochhammer double-loop in which the origin and the singularities (b) are inside one loop, and the point 1 and the singularities (c) are inside the other loop, with the point at infinity and the singularities (a) are outside the contour.

If we employ the second and third expansions of (2.4) for the factors of the integrand associated respectively with (b) and (a), together with

$$(u-x_j)^{-\beta_j}=(u-1)^{-\beta_j}\sum_{m_j=0}^{\infty}\frac{(\beta_j)m_j}{m_j!}\left(\frac{x_j-1}{u-1}\right)^{m_j} \tag{4.2}$$

for the factors associated with (c), then proceeding as in sections 2 and 3, if

$$|x_{p+1}|, \cdots, |x_q|, |x_{q+1}-1|, \cdots, |x_n|, \left|\frac{1}{x_1}\right|, \cdots, \left|\frac{1}{x_p}\right|$$

are all less than unity, we may integrate term-by-term, so that J is proportional to

$$\begin{aligned} & x_1{}^{-\beta_1}\cdots x_p^{-\beta_p}\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(\beta_1)_{m_1}\cdots(\beta)_{m_n}}{m_1!\cdots m_n!} \\ & \times\left(\frac{1}{x_1}\right)^{m_1}\cdots\left(\frac{1}{x_p}\right)^{m_p}(x_{p+1}-1)^{m_{p+1}}\cdots(x_q-1)^{m_q} \\ & \times x_{q+1}^{m_{q+1}}\cdots x_n^{m_n}(-1)^{m_1+\cdots+m_p+m_{q+1}+\cdots+m_n} \\ & \times\int_{[0;1]}(-u)^{\beta_1+\cdots+\beta_q-\gamma+m_1+\cdots+m_p-m_{q+1}-\cdots-m_n} \\ & \times(u-1)^{\gamma-\alpha-\beta_{p+1}-\cdots-\beta_q-m_{p+1}-\cdots-m_q-1}du, \end{aligned} \tag{4.3}$$

and if the inner integral of (4.3) is evaluated using (2.6), the solution under consideration is proportional to

$$\begin{aligned} & x_1^{-\beta_1}\cdots x_p^{-\beta_p}D_{(n)}^{p,q}\Big(\alpha-\beta_1-\cdots-\beta_p, \beta_1, \cdots, \beta_n; \\ & \gamma-\beta_1-\cdots-\beta_q, \alpha+\beta_{p+1}+\cdots+\beta_q-\gamma+1, \\ & \frac{1}{x_1}, \cdots, \frac{1}{x_p}, 1-x_{p+1}, \cdots, 1-x_q, x_{q+1}, \cdots, x_n\Big) \end{aligned} \tag{4.4}$$

where

$$D_{(n)}^{p,q}(a, b_1, \cdots, b_n; c, c'; x_1, \cdots, x_n) \tag{4.5}$$

$$= \sum_{m_1,\cdots,m_n=0}^{\infty} \frac{(a)_{m_{p+1}+\cdots+m_n-m_1-\cdots-m_p}(b_1)_{m_n}}{(c)_{m_{q+1}+\cdots+m_n-m_1-\cdots-m_p}(c')_{m_{p+1}+\cdots+m_q}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}$$

All the solutions of the partial differential system associated with Lauricella's function $F_D^{(n)}$ may thus be expressed in terms of the function $D_{(n)}^{p,q}$, but the total number of distinct soultions obtainable for general n does not appear to be capable of being expressed by means of a simple formpula. The analytical continuation of $F_D^{(n)}$ near any of its singular points may, in general, be expressed in terms of $D_{(n)}^{p,q}$, apart from certain exceptional regions, but the formulae involved become rapidly more complicated as n increases.

It seems appropriate to note that if $p=q=0$, and $q=p$, then $D_{(n)}^{p,q}$ reduces respectively to $F_D^{(n)}$ and the function $C_n^{(p)}$ first defined in [7].

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