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A Necessary and Sufficient Condition for the Existence of a Liapunov Function

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Abstract. In the context of the general theory presented in a previous paper, stability does not necessarily guarantee the existence of a Liapunov function. The main purpose of this paper is to give an exact criterion for the existence of such a function.

Introduction.

In a recent paper by the second author, [2], a general theory of non-asymptotic stability by means of Liapunov type functions was presented. The context was that of a set endowed with a quasiorder (generalizing the semiorbit relation) and two collections of sets, called "quasifilters" (which generalize the notion of neighborhood filters). It was proved that if at least one of the two intervening quasifilters satisfies a certain condition of "admissibility" (essentially, equivalence to a nested countable quasifilter), then stability implies the existence of a (generalized) Liapunov function. On the other hand, an example was given where, in spite of stability, no such function exists. The main result of the present paper is that a Liapunov function exists for a given pair of quasifilters, $(\mathcal{D}, \mathcal{E})$, iff there exists an admissible quasifilter between $\mathcal{D}$ and $\mathcal{E}$ and the system is stable in a somewhat stronger sense than $(\mathcal{D}, \mathcal{E})$-stability.

1. Definitions and principal results of [2].

Let X denote a set, $\mathcal{X}$ the collection of its non-empty subsets, an $\Phi: X \to \mathcal{X}$ a mapping with the property that the relation $y \in \Phi(x)$ is a preorder, i.e. a reflexive, transitive relation. For $\mathcal{A} \subset \mathcal{X}$, define $\Phi(\mathcal{A}) := \{\Phi(A) \mid A \in \mathcal{A}\}$, where $\Phi(A) := \cup \{\Phi(x) | x \in A\}$.

A non-empty subset of $\mathcal{X}$ is called a *quasifilter* on X. Throughout this paper, capital script letters denote quasifilters on X.

We say "$\mathcal{E}$ is *cruder than* $\mathcal{D}$", or $\mathcal{E} \prec \mathcal{D}$, iff

$$\forall E \in \mathcal{E}, \exists D \in \mathcal{D} \quad \text{such that} \quad D \subset E.$$

If both $\mathcal{D} \prec \mathcal{E}$ and $\mathcal{E} \prec \mathcal{D}$ hold, we say $\mathcal{D}$ and $\mathcal{E}$ are *equivalent*, and denote this relation by $\asymp$.

A quasifilter is called *admissible* iff it is equivalent to a quasifilter $\mathcal{B}$ of the form $\mathcal{B} = \{B_i \mid i \in I\}$, where I denotes a subset of $(0, \infty)$, the closure of which contains the origin, and the family of sets $\{B_i\}$ is increasing.

The pair (X, Φ) is called *$(\mathcal{D}, \mathcal{E})$-stable* iff $\mathcal{E} \prec \Phi(\mathcal{D})$ [in other words, iff, given any $E \in \mathcal{E}$, there exists a $D \in \mathcal{D}$ such that $\Phi(D) \subset E$]. We say (X, Φ) is *$\mathcal{E}$-stable* iff it is $(\mathcal{E}, \mathcal{E})$-stable.

Given a function v: $X \to [0, \infty]$, the sets S_v^β $(0 < \beta \leqq \infty)$ and $\mathcal{S}_v$ are defined as follows:

$$S_v^\beta = \{X \in x \mid v(x) < \beta\};$$

$$\mathcal{S}_v = \{S_v^\beta \mid S_v^\beta \neq \phi\}.$$

$\mathcal{S}_v$ is a quasifilter unless v is identically ∞. The function v is called a *Liapunov function for $(X, \Phi, \mathcal{D}, \mathcal{E})$* iff the following conditions are satisfied:

(1) $$\mathcal{E} \prec \mathcal{S}_v;$$

(2) $$\mathcal{S}_v \prec \mathcal{D};$$

(3) $$\Phi(S_v^\beta) \subset S_v^\beta \qquad (\beta > 0).$$

We summarize the three main theorems of [2]:

Theorem A. *If there exists a Liapunov function for $(X, \Phi, \mathcal{D}, \mathcal{E})$, then (X, Φ) is $(\mathcal{D}, \mathcal{E})$-stable.*

This is theorem 1 of [2].

Theorem B. *If (X, Φ) is $(\mathcal{D}, \mathcal{E})$-stable and either $\mathcal{D}$ or $\mathcal{E}$ is admissible, then there exists a Liapunov function for $(X, \Phi, \mathcal{D}, \mathcal{E})$.*

This is the combination of theorems 2 and 3 of [2]; note that in the present paper the reflexivity and transitivity of Φ is a standing assumption.

2. Necessary and sufficient conditions for the existence of a Liapunov function.

We start with the following elementary lemma:

Lemma. *If $\mathcal{E}' \prec \mathcal{E}$ and $\mathcal{D} \prec \mathcal{D}'$, then $(\mathcal{D}, \mathcal{E})$-stability implies $(\mathcal{D}', \mathcal{E}')$-stability.*

We leave the proof to the reader.

We can now formulate the principal result of this paper, which strengthens theorem B and settles the question of existence of Liapunov functions.

Theorem C. *The following condition is necessary and sufficient for the existence of a Liapunov function for $(X, \Phi, \mathcal{D}, \mathcal{E})$:*

There exists an admissible quasifilter $\mathcal{R}$ satisfying the condition

(I) $$\mathcal{E} \prec \mathcal{R} \prec \mathcal{D}$$

and at least one of the following:

(IIa) $$(X, \Phi) \text{ is } (\mathcal{D}, \mathcal{R})\text{-stable};$$

(IIb) (X, Φ) *is* $(\mathcal{R}, \mathcal{E})$*-stable;*

(II c) (X, Φ) *is* $\mathcal{R}$*-stable.*

Moreover, if the function in question exists, one can always find a quasifilter $\mathcal{R}$ *satisfying* (I) *and* (IIc).

Note. As a consequence of the lemma, $\mathcal{R}$-stability implies both $(\mathcal{D}, \mathcal{R})$-*and* $(\mathcal{R}, \mathcal{E})$-stability, and each of these implies $(\mathcal{D}, \mathcal{E})$-stability.

Proof of the theorem. a) Necessity: Suppose v is a Liapunov function for $(X, \Phi, \mathcal{D}, \mathcal{E})$, and put $\mathcal{R}=\mathcal{S}_v$. Then (I) is an immediate consequence of (1) and (2). Moreover, (3) yields $\mathcal{S}_v \prec \Phi(\mathcal{S}_v)$, hence (IIc), which implies the other two conditions.

b) Sufficiency: Suppose $\mathcal{R}$ is admissible and satisfies (I). If (X, Φ) is $(\mathcal{D}, \mathcal{R})$-stable, then, according to theorem B, there exists a Liapunov function v for $(X, \Phi, \mathcal{D}, \mathcal{R})$, thus satisfying the conditions

$$\mathcal{R} \prec \mathcal{S}_v \prec \mathcal{D} \quad \text{and} \quad \Phi(S_v^\beta) \subset S_v^\beta \qquad (\beta>0).$$

These together with (I) imply (1-3), and it follows that v is a Liapunov function for $(X, \Phi, \mathcal{D}, \mathcal{E})$.

If (X, Φ) is $(\mathcal{R}, \mathcal{E})$-stable, theorem B guarantees the existence of a Liapunov function v for $(X, \Phi, \mathcal{R}, \mathcal{E})$, thus satisfying the conditions

$$\mathcal{E} \prec \mathcal{S}_v \prec \mathcal{R} \quad \text{and} \quad \Phi(S_v^\beta) \subset S_v^\beta \qquad (\beta>0),$$

which together with (I) again imply (1-3).

Since $\mathcal{R}$-stability is a special case of the preceding ones, we need not treat it separately. The proof is complete.

3. Example.

Examples in which at least one of the intervening quasifilters is admissible, were given in [2], §7. We will therefore limit ourselves to exhibiting a case in which neither of the quasifilters is admissible, but a Liapunov function exists nevertheless.

Consider a dynamical system on a metric space X, denoting orbits, semi-orbits and limit sets by $\gamma, \gamma^{\pm}, L^{\pm}$, respectively. If Y is a closed, non-empty, proper subset of X, we denote $\mathcal{N}_Y$ the neighborhood filter of Y and introduce the quasifilter

$$\mathcal{N}_Y^*: =\{X \setminus \{x\} \mid x \in X \setminus Y\}$$

which is a subquasifilter of $\mathcal{N}_Y$.

We call Y *topologically prestable* iff it is $(\mathcal{N}_Y, \mathcal{N}_Y^*)$-stable. Explicitly, this means that given any point x not in Y, there exists a neighborhood U of Y such that $\gamma^+(U)$ does not contain x. Except for special cases, neither $\mathcal{N}_Y$ nor

$\mathcal{N}^*_Y$ is admissible. In the case of compact Y, the concept reduces to the notion of prestability (or "Zubov's condition" defined in [2], § 7, II). As we mentioned there, the conditions

(p. 1) $$v>0 \quad \text{on } X\backslash Y$$

and

(p. 2) $$v=0 \text{ and continuous on } Y$$

imply (topological) prestability of Y. This is true independently of whether Y is compact.

Now suppose Y, apart from being closed, invariant and topologically prestable, contains a compact global attractor relative to Y, i. e. a compact set M with the property that all positive semiorbits in Y tend to M; moreover, suppose M contains no positive limit points of orbits outside of Y. [The example given below clearly satisfies these conditions if M is the origin.]

The quasifilter

$$\mathcal{R}=\{R_\beta \mid \beta>0\},$$

where

$$R_\beta=\{x\in X \mid d(\gamma(x), M)<\beta\},$$

is obviously admissible and satisfies the conditions (I) and (IIc) [with $\mathcal{D}=\mathcal{N}_Y$, $\mathcal{E}=\mathcal{N}^*_Y$], thus guaranteeing a Liapunov function in view of theorem C, and indeed

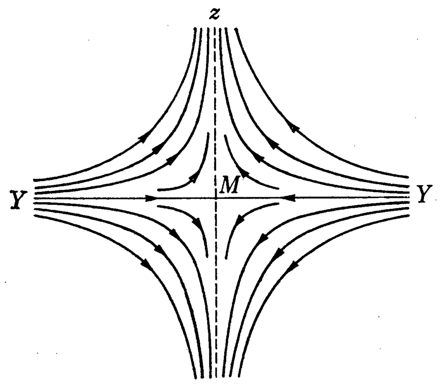
$$v(x) := \inf \{\beta \mid x\in R_\beta\} = d(\gamma(x), M)$$

is such a function. It is easy to see that it satisfies conditions (p. 1) and (p. 2).

As a specific example, satisfying all the hypothesis in question, consider the system of differential equations

$$\dot{y}=-y, \quad \dot{z}=y^2 z,$$

assuming Y to be the y-axis. Here the origin is a "degenerate saddle point", namely a saddle point with its stable separatrix replaced by critical points. The set Y is topologically prestable but fails to satisfy any of the usual definitions of stability (cf. [1], chapt. V).



4. Remark concerning the proof of theorem B.

In [2], the theorems 2 and 3 (now combined into theorem B) were proved separately. We wish to point out that each of them can be obtained easily from the other. We start with the following

Proposition. *In $\mathcal{D}$ is admissible, so is $\Phi(\mathcal{D})$.*

Indeed, if $\mathcal{B}$ is equivalent to $\mathcal{D}$ and satisfies the same conditions as in the definition of admissibility, then these are also satisfied by $\Phi(\mathcal{B})$ which is equivalent to $\Phi(\mathcal{D})$.

Now suppose (X, Φ) is $(\mathcal{D}, \mathcal{E})$-stable and $\mathcal{D}$ is admissible. Then so is $\Phi(\mathcal{D})$, and $(\mathcal{D}, \Phi(\mathcal{D}))$-stability always holds. Now the existence of a Liapunov function v for $(X, \Phi, \mathcal{D}, \Phi(\mathcal{D}))$ follows from theorem 2. This function satisfies the conditions

$$\Phi(\mathcal{D}) \prec \mathcal{S}_v \prec \mathcal{D}.$$

On the other hand, $(\mathcal{D}, \mathcal{E})$-stability implies $\mathcal{E} \prec \Phi(\mathcal{D})$, hence

$$\mathcal{E} \prec \mathcal{S}_v \prec \mathcal{D},$$

which means v is also a Liapunov function for $(X, \Phi, \mathcal{D}, \mathcal{E})$. Thus theorem 3 follows.

This argument, together with the proof of theorem 2 given in [2], yields a shorter proof of theorem 3, hence of theorem B.

References

[1] N. P. Bhatia and G. P. Szegö, Stability Theory of Dynamical Systems, Springer, New York-Heidelberg-Berlin, 1970.

[2] P. Seibert, A unified theory of Liapunov stability, Funkcial. Ekvac., **15** (1972), N° 3.

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