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A Unified Theory of Liapunov Stability

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Abstract

The part of Liapunov's second method concerning (non-asymptotic) stability is presented under minimal hypotheses. Two kinds of inverse theorems are proved. The resulting theory includes most of the important types of Liapunov stability and some others, such as Lagrange stability, as special cases. Each of the four principal cases arising in connection with the inverse theorems is illustrated by a typical example.

Introduction.

It has been recognized for some time, particularly since the appearance of Zubov's monograph [9], that Liapunov's second method assumes its clearest and most elegant form when presented in the context of a continuous dynamical system and in terms of Liapunov functions which do not depend explicitly on the time. However, between this more recent theory and the original "second method", there exists a considerable difference as far as inverse theorems are concerned. Indeed, while the existence of a time-dependent Liapunov function is a consequence of the parallelizability of any flow in state-time space (cf. [6]), this is no longer true for time-independent Liapunov functions. In this paper we present the theory of Liapunov (non-asymptotic) stability under what appear to be the weakest possible conditions: The "flow" is reduced to a set with a quasi-order (i.e. a transitive relation), and stability is defined with respect to two collections of subsets which we call "quasifilters" (a kind of generalized neighborhood filters). In this setting, the theory acquires a sufficient degree of flexibility to allow its adaptation to a wide variety of special situations.

The existence of a Liapunov function requires some restriction on the quasifilters; here we assume that the "final" quasifilter possesses a nested base (with certain reasonable properties). A quasifilter satisfying this condition is called "admissible". Then the existene of a Liapunov function can be proved along the usual lines using a function which, in the classical case, is simply the maximal distance of the positive semiorbit from the set in question. (This idea isa lso used by D. Bushaw in [5]).

In various meaningful concepts of stability, the "final" quasifilter fails to be admissible. For instance, Lagrange stability and the stabilities involving prolongations (cf. [4], Chapt. V, § 4) fall into this class. However, in most of these cases it turns out that the "initial" quasifilter is admissible. In this case we prove the existence theorem by a modified method which, in the classical context, amounts to taking the minimal distance of the negative semiorbit from the set.

Finally, if neither of the quasifilters is admissible, there may not exist a Liapunov function. We illustrate this fact by the example of "topological stability" of a non-compact closed set, which means that both quasifilters are equal to the complete neighborhood filter.

A first version of this theory (without the second inverse theorem) was presented in [8]. There one also finds some additional applications.

1. General definitions and notations.

Let X be a non-empty set, and denote by $\mathfrak{X}$ the set of all non-empty subsets of X. Next, consider a function

$$\Phi\colon X\to\mathfrak{X}.$$

If $A\subset X$, we define

$$\Phi(A)=\bigcup\{\Phi(x)\ |x\in A\}.$$

Any non-empty collection $\mathcal{E}\subset\mathfrak{X}$ will be called a *quasifilter* on X. Let $\mathcal{D}$ and $\mathcal{E}$ be two quasifilters on X, and define:

$$\mathcal{D}\prec\mathcal{E}$$

(read: "$\mathcal{D}$ is *cruder than* $\mathcal{E}$ "or" $\mathcal{E}$ is *finer than* $\mathcal{D}$") if

$$\forall D\in\mathcal{D},\ \exists E\in\mathcal{E} \text{ such that } E\subset D.$$

If both $\mathcal{D}\prec\mathcal{E}$ and $\mathcal{E}\prec\mathcal{D}$, we say that $\mathcal{D}$ and $\mathcal{E}$ are *equivalent*, and denote this relation by $\mathcal{D}\succ\prec\mathcal{E}$.

2. Stability and Liapunov functions.

Given a quasifilter $\mathcal{D}$ on X and a function Φ as above, we define the following quasifilter:

$$\Phi(\mathcal{D})=\{\Phi(D)\ |D\in\mathcal{D}\}.$$

Definition. Let $\mathcal{D}$, $\mathcal{E}$ be two quasifilters on X. Then (X,Φ) is $(\mathcal{D},\mathcal{E})$-*stable* iff $\mathcal{E}\prec\Phi(\mathcal{D})$; in particular, $(\mathcal{E},\mathcal{E})$-stability will be called *stability with respect to* $\mathcal{E}$.

Now consider a function

$$v\colon X\longrightarrow[0,\infty]$$

and define, for each $\beta>0$, the set

$$S_v^\beta=\{x\in X\ |v(x)<\beta\},$$

and for every function v, the collection of sets

$$\mathcal{S}_v=\{S_v^\beta|\ S_v^\beta\neq\emptyset\},$$

which is a quasifilter unless v is identically ∞.

We call v a *Liapunov function* for the quadruplet $(X, \Phi, \mathcal{D}, \mathcal{E})$ iff the following conditions hold:

$(l, 1)$ $$\mathcal{E}\prec\mathcal{S}_v,$$

$(l, 2)$ $$\mathcal{S}_v\prec\mathcal{D},$$

$(l, 3)$ $$(\forall\beta>0)\ \Phi(S_v^\beta)\subset S_v^\beta.$$

The last condition means that v is "non-increasing with respect to Φ", i.e. $y\in\Phi(x)\Longrightarrow v(y)\leqq v(x)$.

3. Sufficient conditions for stability.

Theorem 1. *If there exists a Liapunov function v for $(X, \Phi, \mathcal{D}, \mathcal{E})$, then (X, Φ) is $(\mathcal{D}, \mathcal{E})$-stable.*

Proof. We first show that $(l, 2)$ and $(l, 3)$ together imply

$(l, 2')$ $$\mathcal{S}_v\prec\Phi(\mathcal{D}).$$

Indeed, given $\beta>0$ such that $S_v^\beta\neq\emptyset$, $(l, 2)$ implies:

$$\exists D\in\mathcal{D}\ \text{ such that }\ D\subset S_v^\beta.$$

This, together with $(l, 3)$ [and taking into account that $A\subset B$ obviously implies $\Phi(A)\subset\Phi(B)$], yields

$$\Phi(D)\subset\Phi(S_v^\beta)\subset S_v^\beta,$$

and since $\Phi(D)\in\Phi(\mathcal{D})$ by defintion, $(l, 2')$ follows.

Now, the relation $\prec$ obviously being transitive, $(l, 1)$ and $(l, 2')$ together imply $\mathcal{E}\prec\Phi(\mathcal{D})$, which proves stability.

4. First inverse theorem.

We first introduce an additional axiom concerning the quasifilter $\mathcal{E}$.

A quasifilter $\mathcal{A}$ will be called *admissible* iff it is equivalent (cf. § 1) to a quasifilter $\mathcal{B}$ of the form $\mathcal{B}=\{B_i|i\in I\}$, where I denotes a subset of $(0, \infty)$ the closure of which contains the origin, and $\{B_i\}$ is monotone increasing.

Theorem 2. *If the relation $y\in\Phi(x)$ is reflexive and transitive, and $\mathcal{E}$ is admissible, then $(\mathcal{D}, \mathcal{E})$-stability of (X, Φ) implies the existence of a Liapunov* *function v for $(X, \Phi, \mathcal{D}, \mathcal{E})$.*

Proof. Suppose $\mathcal{E}$ is equivalent to $\mathcal{B}$. Then define the function v as follows [using the notation of the preceding definition]:

$$(1) \qquad v(x) = \inf\{j \in I \mid \Phi(x) \subset B_j\}.$$

If $\Phi(x)$ is not contained in any B_j, put $v(x) = \infty$.

We first verify $(l, 2)$: Let $\beta > 0$; then, because of $0 \in \bar{I}$, there exists an $i \in I$ such that $i < \beta$. Since $\mathcal{B} \prec \mathcal{E}$, $\exists E \in \mathcal{E}$ such that $E \subset B_i$. Now stability implies

$$\exists D \in \mathcal{D} \text{ such that } \Phi(D) \subset E \subset B_i,$$

and then (1) yields $v \leqq i < \beta$ on D, or $D \subset S_v^\beta$. Since $\mathcal{S}_v$ obviously contains only sets S_v^β with $\beta > 0$, $(l, 2)$ follows. As an incidental result, we have

$$(2) \qquad \beta > 0 \Longrightarrow S_v^\beta \neq \emptyset.$$

$(l, 1)$ is equivalent to

$$\forall E \in \mathcal{E}, \ \exists j \in I \text{ such that } \emptyset \neq S_v^j \subset E.$$

Because of $\mathcal{E} \prec \mathcal{B}$, there exists a $j \in I$ such that

$$(3) \qquad B_j \subset E.$$

Moreover, for any $x \in S_v^j$, $v(x) < j$, and hence (1) implies $\Phi(x) \subset B_j$. Now reflexivity of Φ and (3) yield $x \in \Phi(x) \subset E$, hence $\emptyset \neq S_v^j \subset E$ [$S_v^j \neq \emptyset$ being implied by (2)].

Proof of $(l, 3)$: Suppose $y \in \Phi(x)$; then transitivity of Φ implies $\Phi(y) \subset \Phi(x)$, and now (1) yields $v(y) < v(x)$.

This is equivalent to $(l, 3)$.

5. Second inverse theorem.

If the relation $y \in \Phi(x)$ is reflexive, then Φ possesses an inverse, Φ^*, i.e. a function

$$\Phi^*: X \longrightarrow \mathfrak{X}$$

such that

$$x \in \Phi(y) \Longleftrightarrow y \in \Phi^*(x).$$

Moreover, if one of these relations is transitive (reflexive), so is the other.

If $A \subset X$, we write A^* for $X \setminus A$.

Lemma. *If the relation defined by Φ is reflexive,*

$$\Phi(D) \subset E \Longleftrightarrow \Phi^*(E^*) \subset D^*.$$

The proof is elementary and is therefore left to the reader.

We may now formulate the following theorem which complements the preceding one.

Theorem 3. *If Φ is reflexive and transitive, and $\mathcal{D}$ is admissible, then $(\mathcal{D}, \mathcal{E})$-stability of (X, Φ) implies the existence of a Liapunov function v for $(X, \Phi, \mathcal{D}, \mathcal{E})$.*

Proof. Let $\mathcal{B}$ be as in the definition of admissibility and define the function v as follows:

$$v(x) = \sup \{j \in I | \Phi(x) \cap B_j = \emptyset\}. \tag{4}$$

If $\Phi^*(x)$ intersects all B_j's, put $v(x) = 0$.

Proof of $(l, 2)$: Given $\beta > 0$, choose $i \in I$ such that $i < \beta$; since Φ (and hence Φ^*) is reflexive, $\Phi^*(x)$ trivially intersects B_i for any $x \in B_i$. Thus $v \leqq i < \beta$ on B_i, or $B_i \subset S_v^\beta$. Since $B_i \in \mathcal{B} \prec \mathcal{D}$, it follows that $\mathcal{S}_v \prec \mathcal{D}$. This is $(l, 2)$. Moreover, since $\beta > j > 0$ implies $S_v^j \subset S_v^\beta$, (2) again holds.

Proof of $(l, 1)$: Given $E \in \mathcal{E}$, choose $D \in \mathcal{D}$ such that $\Phi(D) \subset E$; due to the lemma, this is equivalent to $\Phi^*(E^*) \subset D^*$ or

$$x \in E^* \Longrightarrow \Phi^*(x) \cap D = \emptyset. \tag{5}$$

Next, choose $j \in I$ such that $B_j \subset D$; then, for $x \in E^*$, (5) implies $\Phi^*(x) \cap B_j = \emptyset$, and now (4) yields $v(x) \geqq j$; consequently, $E^* \subset S_v^{j^*}$, which is equivalent to $S_v^j \subset E$. Moreover, $S_v^j \neq \emptyset$, due to (2), i.e. $S_v^j \in \mathcal{S}_v$ and $(l, 1)$ follows.

Finally, transitivity of Φ (and hence of Φ^*) implies $y \in \Phi(x) \Longleftrightarrow x \in \Phi^*(y)$ $\Longrightarrow \Phi^*(x) \subset \Phi^*((\Phi^*(y)) \subset \Phi^*(y) \Longrightarrow [\Phi^*(y) \cap B_j = \emptyset \Longrightarrow \Phi^*(x) \cap B_j = \emptyset]$, and now (4) yields

$$y \in \Phi(x) \Longrightarrow v(y) \leqq v(x),$$

which proves $(l, 3)$.

6. Boundedness.

We introduce the following definitions:

6.1. $\mathcal{D} \succ\!\!\prec \mathcal{E}$ iff $\forall D \in \mathcal{D}$, $\exists E \in \mathcal{E}$ such that $D \subset E$.

6.2. $\mathcal{D}^* = \{D^* | D \in \mathcal{D}\}$ (and $\mathcal{E}^*$ analogously).

Obviously,

$$\mathcal{D} \succ\!\!\prec \mathcal{E} \Longleftrightarrow \mathcal{D}^* \prec E^*.$$

Suppose Φ is reflexive; then we say that (X, Φ) is $(\mathcal{D}, \mathcal{E})$-*bounded* iff $\Phi(\mathcal{D}) \succ\!\!\prec \mathcal{E}$.

Proposition ["*Duality principle*"]. *(X, Φ) is $(\mathcal{D}, \mathcal{E})$-bounded iff (X, Φ^*) is*

$(\mathcal{E}^*, \mathcal{D}^*)$*-stable.*

Proof: $\Phi(\mathcal{D}) \succ \mathcal{E} \Longleftrightarrow [\forall D \in \mathcal{D}, \ \exists E \in \mathcal{E}$ such that $\Phi(D) \subset E] \underset{\text{Lemma}}{\Longleftrightarrow}$ $[\forall D^* \in \mathcal{D}^*, \exists E^* \in \mathcal{E}^*$ such that $\Phi^*(E^*) \subset D^*] \Longleftrightarrow \mathcal{D} \prec \Phi^*(\mathcal{E}^*)$. Q.E.D.

As a consequence, to each of the preceding theorems on stability there corresponds an analogous theorem on boundedness. (Compare the "duality principle" in [2], § 6).

7. Examples.

Let there be given a topological space X, a transitive and reflexive map Φ: $X \longrightarrow \mathfrak{X}$, and a pair of quasifilters $\mathcal{D}$, $\mathcal{E}$ on X. Then the inverse theorems 2 and 3 suggest the consideration of the following four cases:

I) $\mathcal{E}$ *is admissible.* In this case the first inverse theorem can be applied. An example where $\mathcal{E}$ is admissible while $\mathcal{D}$ is not, is furnished by stability (in the sense of Bhatia [3]; c.f. also [4]) of a closed subset M of a metric space X. Here $\mathcal{D}$ is the complete neighborhood filter $\mathfrak{N}_M$ of M, and $\mathcal{E}$ is the system of its δ-neighborhoods ($\delta>0$), $\mathfrak{M}_M$. One verifies easily that our theorems 1 and 2, in this case, yield theorem 4.5 of [4], Chapt. V. As a special case, this includes non-uniform stability in a non-autonomous differential system.

II) $\mathcal{D}$ *is admissible.* In this case our second inverse theorem can be applied. As an example, consider Zubov's necessary condition for stability or, as we will also call it, "prestability" of a compact set $M \subset X$, which is defined as follows:

$$\forall x \notin M, \ \exists U \in \mathfrak{N}_M \text{ such that } x \notin \Phi(U)$$

[or, equivalently, $\Phi^*(x) \cap U = \emptyset$]. In the case of a dynamical system, this is clearly equivalent to the condition that M contains no negative limit points of any orbit not contained in M].

If $\mathfrak{N}'_M$ denotes the quasifilter

$$\mathfrak{N}'_M := \{X - \{x\} \mid x \notin M\}$$

(which is a sub-quasifilter of $\mathfrak{N}_M$), then prestability reduces to $(\mathfrak{N}_M, \mathfrak{N}'_M)$-stability. If X is metric, the first quasifilter is admissible, while the second is not. As one verifies easily, the conditions $(l, 1)$ and $(l, 2)$ assume the following form:

(s, 1) $$x \notin M \Longrightarrow v(x) > 0;$$

(s, 2) $$v(x_n) \longrightarrow 0 \text{ as } x_n \longrightarrow M \text{ or, more succinctly,}$$

$$\bar{v} = 0 \text{ on } M,$$

where $\bar{v}$ denotes the upper limit function of v: $\bar{v}(x) := \overline{\lim_{y \to x}}\, v(y)$. Hence these conditions, together with the monotonicity condition $(l, 3)$, imply prestability. Because of theorem 3, they are also necessary. {This criterion was first stated by R. Acosta A. in [1]. (He uses the term "semistable" rather than "prestable"). In [4], it is given, erroneously, as a condition for equistability (Chapt. V, theorem 4.7).} One easily verifies that (supposing X metric) the function (4) assumes the particularly simple form

$$v(x)=d(\Phi^*(x), M).$$

It is interesting to observe that Zubov's condition, or "prestability", is connected by duality with the classical concept of Lagrange stability. (X, Φ) is said to be (positively) Lagrange stable iff, for every $x\in X$, $\overline{\Phi(x)}$ is compact. If $\mathcal{U}$ denotes the "ultraquasifilter" on X {"Ultra", because $\mathcal{U}$ is finer than any quasifilter on X}, i.e. the quasifilter consisting of all one-point sets, $\{x\}$, $x\in X$, and $\mathcal{C}$ the quasifilter consisting of all compact subsets of X, then Lagrange stability reduces to $(\mathcal{U}, \mathcal{C})$-boundedness. Furthermore, if $X' := X\cup\{\xi\}$ is the one-point compactification of X, (X, Φ) is Lagrange stable iff ξ is prestable with respect to Φ^*, due to the "duality principle". This, together with our criterion for prestability (and passing from v to 1/v), yields the unnumbered theorem on p. 113 of [7], along with its inverse. Moreover, by applying this criterion to subsets, one easily gets estimates for the *region of Lagrange stability*.

III) *$\mathcal{D}$ and $\mathcal{E}$ are both admissible.* In this case (which contains the classical one where $\mathcal{D}$ and $\mathcal{E}$ are both equal to the same admissible neighborhood filter), one may apply either one of the two inverse theorems. As a non-trivial example, let X be a metric space, $N\subset X$ closed, and $M\subset N$ compact; then consider the concept of $(\mathcal{N}_M, \mathcal{M}_N)$-stability (with the notations of I)), i.e., given any δ-neighborhood U of M, there exists a neighborhood V of N such that $\Phi(V)\subset U$. $\mathcal{N}_M$ and $\mathcal{M}_N$ are both admissible. Note that, when M is compact, $\mathcal{N}_M=\mathcal{M}_M$. The functions (1) and (4) assume the following forms, respectively:

$$v_1(x)=\sup\{d(y, N) \mid y\in\Phi(x)\},$$

$$v_2(x)=\inf\{d(y, M) \mid y\in\Phi^*(x)\}.$$

Each of them satisfies the conditions $(l, 1\text{–}3)$.

IV) *Neither $\mathcal{D}$ nor $\mathcal{E}$ is admissible.* In this case, our theory is not applicable. An example is furnished by "*topological stability*" of a closed set $M\subset X$, i.e., stability with respect to the complete neighborhood filter, $\mathcal{N}_M$, of M. If X is metric, M closed, non-compact, and $X\setminus M$ is dense on X, then it is not difficult to see that no Liapunov function exists. Indeed, if v were such a function, the conditions $(l, 1)$, $(l, 2)$ would reduce to

$$\mathcal{N}_M \succ \mathcal{S}_v. \qquad (6)$$

$\mathcal{S}_v \prec \mathcal{N}_M$ implies

(7) $$\bar{v}=0 \text{ on } M.$$

On the other hand, we may choose a sequence $(x_n) \subset M$ without accumulation points. Since $X \backslash M$ is dense on X, all sets $B_\varepsilon(x_n) \backslash M$ $[B_\varepsilon(x):=\{y \in X | d(x, y)<\varepsilon\}]$ with $\varepsilon>0$, are non-empty; this together with (7) yields a sequence $(y_n) \subset B_{1/n}(x_n) \backslash M$ such that

(8) $$v(y_n)<\frac{1}{n}.$$

Obviously, (y_n) has no accumulation points either, and so $X \backslash (y_n)$ is a neighborhood of M which, because of (8), does not contain any sets of $\mathcal{S}_v$. Hence $\mathcal{N}_M \not\prec \mathcal{S}_v$, which means that it is impossible to satisfy (6).

8. Relation to the continuity of multiple-valued selfmappings.

The rather natural idea of regarding stability as a kind of continuity, while being quite obvious in the classical context, becomes somewhat obscured by the abstract point of view adopted in this paper. However, it can easily be retrieved on a higher level of generality by passing to the product space $Z=X \times X$ and considering the multivalued selfmapping Ψ of Z defined by

$$\Psi(x, y):=\Phi(x) \times \Phi(y).$$

Our theory can then be transferred verbatim from (X, Φ) to (Z, Ψ), thus yielding a theory of proximality type conditions which contains Liapunov stability (in the sense in which we have defined it) as a special case. More concretely, if $\mathcal{F}$ and $\mathcal{G}$ are quasifilters on Z, then $(\mathcal{F}, \mathcal{G})$-stability (in the sense of the previous definition) will conveniently be called "$(\mathcal{F}, \mathcal{G})$-proximality" since in the classical case it means that orbits which start close together remain close together. This concept of proximality can be constructed as a generalized concept of continuity of the mapping Ψ.

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