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Existence Theorems of Quasiperiodic Solutions to Nonlinear Differential Systems

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Abstract. In his previous paper [2], the author showed that pseudoperiodic differential operators and quasiperiodic differential operators have Green functions under some conditions. On the basis of these results, in the present paper, the author establishes a series of existence theorems of pseudoperiodic solutions to nonlinear pseudoperiodic differential systems and those of quasiperiodic solutions to nonlinear quasiperiodic differential systems.

1. Introduction.

In his pervious paper [2], the author considered a matrix-valued function $f(t, u)=f(t, u_1, u_2, \cdots, u_m)$ of real variables t and $u=(u_1, u_2, \cdots, u_m)$ such that it is periodic in $u_1, u_2, \cdots, u_m$ with periods $\omega_1, \cdots, \omega_m$ and in addition it satisfies the equality

$$(1.1) \qquad f(t+\omega_0, u)=f(t, u+\omega_0)=f(t, u_1+\omega_0, u_2+\omega_0, \cdots, u_m+\omega_0).$$

He called such a function $f(t, u)$ a *pseudoperiodic function of t and u with periods ω_0 and $\omega=(\omega_1, \omega_2, \cdots, \omega_m)$.*

Throughout the paper it will be assumed that

$$(1.2) \qquad \omega_i>0 \qquad (i=0, 1, 2, \cdots, m).$$

Let $A(t, u)$ be a continuous square matrix dependent on t and u and pseudoperiodic in t and u with periods ω_0 and ω, and let L be a pseudoperiodic differential operator defined by

$$(1.3) \qquad Ly=\frac{dy}{dt}-A(t, u)y.$$

In [2] the author called the operator L to be *regular* if there is a continuous square matrix $P(u)=P(u_1, u_2, \cdots, u_m)$ periodic in $u_1, u_2, \cdots, u_m$ with periods $\omega_1, \omega_2, \cdots, \omega_m$ satisfying the conditions as follows:

$$(1.4) \qquad P^2(u)=P(u),$$

$$(1.5) \qquad \begin{cases} \|\Phi(t, u)P(u)\| \leqq K_0 e^{-\gamma t} & \text{for } t \geqq 0, \\ \|\Phi(t, u)[E-P(u)]\| \leqq K_0 e^{\gamma t} & \text{for } t \leqq 0, \end{cases}$$

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(1.6) $$P(u+\omega_0)\Phi(\omega_0, u)=\Phi(\omega_0, u)P(u),$$

where E is the unit matrix, $\Phi(t, u)$ is the fundamental matrix of the differential system

(1.7) $$Ly=0$$

satisfying the initial condition

(1.8) $$\Phi(0, u)=E,$$

K_0 and γ are positive numbers, and $\|\cdot\|$ denotes any norm.

In [2] it was shown that *if the pseudoperiodic differential operator L is regular, then*

1° *L has a Green function $G(t, s, u)$ with the property*

(1.9) $$\|G(t, s, u)\|\leqq Ke^{-\gamma|t-s|} \quad (K, \gamma>0) \quad \textit{for all } t, s \textit{ and } u,$$

2° *for any continuous vector-valued function $f(t, u)$ of real variables t and u bounded for all t the differential system*

(1.10) $$Lx=f(t, u)$$

has a unique solution $x=x(t, u)$ bounded for all t and it is given by

(1.11) $$x(t, u)=\int_{-\infty}^{\infty} G(t, s, u)f(s, u)ds,$$

3° *for any continuous vector-valued pseudoperiodic function $f(t, u)$ with periods ω_0 and ω the differential system* (1.10) *has a unique pseudoperiodic solution $x=x(t, u)$ with periods ω_0 and ω, and it is given by* (1.11).

In [2] the author called a matrix-valued function $f(t)$ of a real variable t a *quasiperiodic function with periods $\omega_0, \omega_1, \cdots, \omega_m$* if

(1.12) $$f(t)=\bar{f}(t, 0)$$

for some continuous matrix-valued function $\bar{f}(t, u)$ pseudoperiodic in t and u with periods ω_0 and $\omega=(\omega_1, \omega_2, \cdots, \omega_m)$. In what follows, a continuous pseudoperiodic function $\bar{f}(t, u)$ corresponding to a given quasiperiodic function $f(t)$ in the above way will be called briefly a *continuous pseudoperiodic function corresponding to a given quasiperiodic function $f(t)$*. As was shown in [2], the condition (1.12) is equivalent to the condition that

(1.13) $$f(t)=f^{(0)}(t, t, t, \cdots, t)$$

for some continuous matrix-valued function $f^{(0)}(t, u)=f^{(0)}(t, u_1, u_2, \cdots, u_m)$ of real variables $t, u_1, u_2, \cdots, u_m$ periodic in these variables with periods $\omega_0, \omega_1, \omega_2, \cdots, \omega_m$.

Let $A(t)$ be a square matrix quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$

and $\bar{A}(t, u)$ be a continuous pseudoperiodic matrix corresponding to $A(t)$. In [2] the quasiperiodic differential operator L defined by

(1.14) $$Ly=\frac{dy}{dt}-A(t)y$$

was called to be *regular* if and only if the corresponding pseudoperiodic differential operator $\bar{L}$ defined by

(1.15) $$\bar{L}y=\frac{dy}{dt}-\bar{A}(t, u)y$$

is regular.

Concerning quasiperiodic differential operators, in [2] it was shown from the results concerning pseudoperiodic differential operators that *if the quasiperiodic differential operator L defined by* (1.14) *is regular, then*

1° *L has a Green function* $G(t, s)=\bar{G}(t, s, 0)$ *with the property*

(1.16) $$\|G(t, s)\| \leqq Ke^{-\gamma|t-s|} \quad (K, \gamma>0) \text{ for all } t \text{ and } s,$$

where $\bar{G}(t, s, u)$ *is a Green function of the pseudoperiodic differential operator corresponding to* L,

2° *for any quasiperiodic vector-valued function* $f(t)$ *with periods* $\omega_0, \omega_1, \cdots, \omega_m$ *the differential system*

(1.17) $$Lx=f(t)$$

has a unique quasiperiodic solution $x=x(t)$ *with periods* $\omega_0, \omega_1, \cdots, \omega_m$, *and it is given by*

(1.18) $$x(t)=\int_{-\infty}^{\infty} G(t, s)f(s)ds.$$

In the present paper, first, from the results of [2] mentioned above two existence theorems of pseudoperiodic solutions to nonlinear pseudoperiodic differential systems will be established. The first one will be concerned with the differential system of the form

(1.19) $$\frac{dx}{dt}=A(t, u)x+f(t, u, x)$$

and the second one will be a theorem of the same character as Proposition 3 of the paper [1], that is, a theorem which enables one to know the existence of an exact pseudoperiodic solution from that of a pseudoperiodic approximate solution and in addition enables one to know an error bound of the approximate solution taken into account.

Next, on the basis of these theorems, two corresponding existence theorems of quasiperiodic solutions to nonlinear quasiperiodic differential systems will be established. In establishing these theorems it will be assumed that *reciprocals of the periods $\omega_0, \omega_1, \cdots, \omega_m$ of quasiperiodic functions appearing in the theorems are rationally linearly independent, that is,*

$$\sum_{i=0}^{m} \frac{r_i}{\omega_i} \neq 0 \tag{1.20}$$

for any rational numbers $r_0, r_1, \cdots, r_m$ except $r_0=r_1=\cdots=r_m=0$. As will be shown in **3**, no generality is lost by this assumption. Lastly from the second theorem concerning quasiperiodic solutions will be proved a theorem concerning perturbations of quasiperiodic differential systems.

2. Pseudoperiodic solutions to nonlinear pseudoperiodic differential systems.

Theorem 1. *Given a pseudoperiodic differential system of the form*

$$\frac{dx}{dt}=A(t,u)x+f(t,u,x), \tag{2.1}$$

where $A(t,u)$ is a continuous matrix pseudoperiodic in t and u with periods ω_0 and ω, and $f(t,u,x)$ is a continuous vector-valued function pseudoperiodic in t and u with periods ω_0 and ω for all $x \in D$. Here D is a region of the x-space.

Suppose that the pseudoperiodic differential operator L defined by

$$Ly=\frac{dy}{dt}-A(t,u)y \tag{2.2}$$

is regular. Let its Green function $G(t,s,u)$ satisfy

$$\|G(t,s,u)\| \leqq Ke^{-\gamma|t-s|} \quad \text{for all } t, s, \text{ and } u, \tag{2.3}$$

where K and γ are positive numbers. Assume that $f(t,u,x)$ satisfies the condition

$$\|f(t,u,x')-f(t,u,x'')\| \leqq \frac{\kappa}{M}\|x'-x''\| \tag{2.4}$$

for all t and u and any $x', x'' \in D$, where

$$M=\frac{2K}{\gamma} \tag{2.5}$$

and κ is a non-negative number smaller than unity.

If there is a continuous vector-valued function $x_0(t,u)$ pseudoperiodic in t and u with periods ω_0 and ω such that

$$x_0(t,u) \in D \quad \text{for all } t \text{ and } u, \tag{2.6}$$

and

$$S=\left\{x \,\middle|\, \|x-x_1(t,u)\| \leqq \frac{\kappa}{1-\kappa}\|x_1-x_0\|_n \text{ for some } t \text{ and } u\right\} \subset D, \tag{2.7}$$

then the given system (2.1) *has a unique solution* $x=\hat{x}(t,u)$ *pseudoperiodic in* t *and* u *with periods* ω_0 *and* ω *such that*

$$\hat{x}(t,u)\in D \quad \text{for all } t \text{ and } u, \tag{2.8}$$

and moreover for $\hat{x}(t,u)$ *it holds that*

$$\|\hat{x}(t,u)-x_0(t,u)\|\leqq\frac{1}{1-\kappa}\|x_1-x_0\|_n. \tag{2.9}$$

In (2.7), *however,*

$$x_1(t,u)=\int_{-\infty}^{\infty}G(t,s,u)f[s,u,x_0(s,u)]ds \tag{2.10}$$

and $\|f\|_n$ *denotes* $\sup_{t,u}\|f(t,u)\|$ *for any vector-valued function bounded for all* t *and* u.

Proof. Consider the iterative process

$$x_{n+1}(t,u)=\int_{-\infty}^{\infty}G(t,s,u)f[s,u,x_n(s,u)]ds \quad (n=0,1,2,\cdots). \tag{2.11}$$

By the induction we shall prove that this iterative process can be continued indefinitely and that

$$\|x_{n+1}-x_n\|_n\leqq\kappa^n\|x_1-x_0\|_n \quad (n=0,1,2,\cdots), \tag{2.12}$$

$$x_{n+1}(t,u)\in S \quad \text{for all } t \text{ and } u \quad (n=0,1,2,\cdots). \tag{2.13}$$

For $n=0$, (2.12) and (2.13) is evident. Assume that (2.12) and (2.13) are valid up to $n-1$. Then from (2.11) we have

$$x_{n+1}(t,u)-x_n(t,u)=\int_{-\infty}^{\infty}G(t,s,u)\{f[s,u,x_n(s,u)]-f[s,u,x_{n-1}(s,u)]\}ds,$$

therefore by (2.3), (2.4) and (2.5) we have

$$\begin{aligned}\|x_{n+1}(t,u)-x_n(t,u)\|&\leqq\left[\int_{-\infty}^{t}Ke^{-\gamma(t-s)}ds+\int_{t}^{\infty}Ke^{-\gamma(s-t)}ds\right]\times\frac{\kappa}{M}\|x_n-x_{n-1}\|_n\\&=\frac{2K}{\gamma}\cdot\frac{\kappa}{M}\|x_n-x_{n-1}\|_n\\&=\kappa\|x_n-x_{n-1}\|_n,\end{aligned}$$

from which follows

$$\|x_{n+1}-x_n\|_n\leqq\kappa\|x_n-x_{n-1}\|_n. \tag{2.14}$$

Then by the assumption of the induction we have (2.12) for n. Then since

$$\|x_{n+1}-x_1\|_n\leqq\|x_{n+1}-x_n\|_n+\|x_n-x_{n-1}\|_n+\cdots+\|x_2-x_1\|_n,$$

it follows from (2.12) that

$$(2.15)\qquad \|x_{n+1}-x_1\|_n \leqq (\kappa^n+\kappa^{n-1}+\cdots+\kappa)\|x_1-x_0\|_n$$
$$\leqq \frac{\kappa}{1-\kappa}\|x_1-x_0\|_n,$$

which proves (2.13) for n.

By (2.13) it is clear that the iterative process (2.11) can be continued indefinitely. Hence we have an infinite sequence $\{x_n(t,u)\}$, which by (2.12) is uniformly convergent for all t and u. Hence we have a continuous vector-valued function $\hat{x}(t,u)$ such that

$$(2.16)\qquad \hat{x}(t,u)=\lim_{n\to\infty} x_n(t,u).$$

Now since $x_0(t,u)$ is pseudoperiodic in t and u with periods ω_0 and ω, as is seen from the results of [2] stated in **1**, $x_1(t,u)$ given by (2.10) is also pseudoperiodic in t and u with periods ω_0 and ω and $x=x_1(t,u)$ is a solution to the differential system

$$Lx=f[t,u,x_0(t,u)].$$

Continuing the same reasoning, we then see that $x_n(t,u)$ obtained by the iterative process (2.11) are all pseudoperiodic in t and u with periods ω_0 and ω. Then by (2.16), we easily see that $\hat{x}(t,u)$ is also pseudoperiodic in t and u with periods ω_0 and ω.

Now let $n\to\infty$ in (2.15). Then by (2.16) we have

$$(2.17)\qquad \|\hat{x}-x_1\|_n \leqq \frac{\kappa}{1-\kappa}\|x_1-x_0\|_n,$$

which clearly implies

$$\hat{x}(t,u)\in S\subset D \qquad \text{for all } t \text{ and } u.$$

Then, since

$$\left\|\hat{x}(t,u)-\int_{-\infty}^{\infty}G(t,s,u)f[s,u,\hat{x}(s,u)]ds\right\|$$
$$\leqq \|\hat{x}(t,u)-x_{n+1}(t,u)\|$$
$$+\left\|\int_{-\infty}^{\infty}G(t,s,u)\{f[s,u,x_n(s,u)]-f[s,u,\hat{x}(s,u)\}ds\right\|,$$

similarly to (2.14) we have

$$\left\|\hat{x}(t,u)-\int_{-\infty}^{\infty}G(t,s,u)f[s,u,\hat{x}(s,u)]ds\right\|$$
$$\leqq \|\hat{x}(t,u)-x_{n+1}(t,u)\|+\kappa\|x_n-\hat{x}\|_n.$$

Then letting $n\to\infty$, from (2.16) we have

$$\left\|\hat{x}(t,u)-\int_{-\infty}^{\infty}G(t,s,u)f[s,u,\hat{x}(s,u)]ds\right\|=0,$$

that is,

$$\hat{x}(t,u)=\int_{-\infty}^{\infty}G(t,s,u)f[s,u,\hat{x}(s,u)]ds. \tag{2.18}$$

This implies that $x=\hat{x}(t,u)$ satisfies the differential system

$$Lx=f[t,u,\hat{x}(t,u)],$$

that is, $x=\hat{x}(t,u)$ satisfies the given differential system (2.1).

Now, since

$$\|\hat{x}-x_0\|_n\leqq\|\hat{x}-x_1\|_n+\|x_1-x_0\|_n,$$

by (2.17) we have

$$\|\hat{x}-x_0\|_n\leqq\frac{1}{1-\kappa}\|x_1-x_0\|_n,$$

which proves (2.9).

It now remains to prove the uniqueness of pseudoperiodic solutions to (2.1) in D. Let $x=\hat{x}'(t,u)$ be an arbitrary solution to (2.1) pseudoperiodic in t and u with periods ω_0 and ω such that

$$\hat{x}'(t,u)\in D \qquad \text{for all } t \text{ and } u.$$

Then $x=\hat{x}'(t,u)$ satisfies the differential system

$$Lx=f[t,u,\hat{x}'(t,u)].$$

Therefore by the results of [2] stated in **1**, we have

$$\hat{x}'(t,u)=\int_{-\infty}^{\infty}G(t,s,u)f[s,u,\hat{x}'(s,u)]ds. \tag{2.19}$$

Then subtracting (2.18) from (2.19) side by side, similarly to (2.14) we have

$$\|\hat{x}'-\hat{x}\|_n\leqq\kappa\|\hat{x}'-\hat{x}\|_n,$$

which implies

$$\|\hat{x}'-\hat{x}\|_n=0 \tag{2.20}$$

since $0\leqq\kappa<1$. Equality (2.20) implies

$$\hat{x}'(t,u)\equiv\hat{x}(t,u),$$

which proves the uniqueness of pseudoperiodic solutions to (2.1) in D.

Q.E.D.

In Theorem 1, suppose that the region D is the whole x-space. Then the condition concerned with $x_0(t, u)$ is fulfilled automatically. Hence we have

Corollary. *In* (2.1), *suppose that* $A(t, u)$ *is a continuous matrix pseudoperiodic in* t *and* u *with periods* ω_0 *and* ω, *and* $f(t, u, x)$ *is a continuous vector-valued function pseudoperiodic in* t *and* u *with periods* ω_0 *and* ω *for all* x.

If the pseudoperiodic differential operator L *defined by* (2.2) *is regular and* $f(t, u, x)$ *satisfies the condition* (2.4) *for all* x' *and* x'', *then the given system* (2.1) *has a unique solution* $x=\hat{x}(t, u)$ *pseudoperiodic in* t *and* u *with periods* ω_0 *and* ω.

The following theorem is an extension of Proposition 3 of [1].

Theorem 2. *Given a nonlinear pseudoperiodic differential system*

$$\frac{dx}{dt}=X(t, u, x), \tag{2.21}$$

where $X(t, u, x)$ *is a vector-valued function pseudoperiodic in* t *and* u *with periods* ω_0 *and* ω *and continuously differentiable with respect to* x *for* $x \in D$. *Here* D *is a region of the* x*-space.*

Suppose that the differential system (2.21) *has an approximate solution* $x=x_0(t, u)$ *pseudoperiodic in* t *and* u *with periods* ω_0 *and* ω *such that*

$$x_0(t, u) \in D \qquad \text{for all } t \text{ and } u, \tag{2.22}$$

$dx_0(t, u)/dt$ *is continuous in* t *and* u *for all* t *and* u, *and*

$$\left\| \frac{dx_0(t, u)}{dt} - X[t, u, x_0(t, u)] \right\| \leqq r \qquad \text{for all } t \text{ and } u. \tag{2.23}$$

Further suppose that there are a positive number δ, *a non-negative number* $\kappa<1$ *and a continuous matrix* $A(t, u)$ *pseudoperiodic in* t *and* u *with periods* ω_0 *and* ω *satisfying the conditions as follows* :

1° *the pseudoperiodic differential operator* L *defined by*

$$Ly=\frac{dy}{dt}-A(t, u)y \tag{2.24}$$

is regular,

2°

$$\begin{cases} \text{(i)} \quad D_\delta=\{x \mid \|x-x_0(t, u)\|<\delta \quad \text{for some } t \text{ and } u\} \subset D, \\ \text{(ii)} \quad \|\Psi(t, u, x)-A(t, u)\| \leqq \dfrac{\kappa}{M} \quad \text{for any } (t, u, x) \text{ satisfying} \\ \qquad\qquad \|x-x_0(t, u)\|<\delta, \\ \text{(iii)} \quad \dfrac{Mr}{1-\kappa}<\delta. \end{cases} \tag{2.25}$$

Here $\Psi(t, u, x)$ *is the Jacobian matrix of* $X(t, u, x)$ *with respect to* x *and*

$$M=\frac{2K}{\gamma} \tag{2.26}$$

where K *and* γ *are positive numbers such that the Green function* $G(t,s,u)$ *of* L *satisfies*

$$\|G(t,s,u)\| \leqq Ke^{-\gamma|t-s|} \quad \text{for all } t, s, \text{ and } u. \tag{2.27}$$

The given differential system (2.21) *then has a solution* $x=\hat{x}(t,u)$ *pseudoperiodic in* t *and* u *with periods* ω_0 *and* ω *such that*

$$\|\hat{x}-x_0\|_n \leqq \frac{Mr}{1-\kappa} < \delta. \tag{2.28}$$

Moreover a solution $x=\hat{x}(t,u)$ *to* (2.21) *pseudoperiodic in* t *and* u *with periods* ω_0 *and* ω *satisfying the inequality*

$$\|\hat{x}(t,u)-x_0(t,u)\| < \delta \quad \text{for all } t \text{ and } u \tag{2.29}$$

is unique.

Proof. Write the given system (2.21) in the form

$$\frac{dx}{dt}=A(t,u)x+f(t,u,x), \tag{2.30}$$

where

$$f(t,u,x)=X(t,u,x)-A(t,u)x. \tag{2.31}$$

Put

$$\frac{dx_0(t,u)}{dt}=X[t,u,x_0(t,u)]+\eta(t,u), \tag{2.32}$$

then clearly $\eta(t,u)$ is continuous and is pseudoperiodic in t and u with periods ω_0 and ω, and by (2.23) it satisfies the inequality

$$\|\eta(t,u)\| \leqq r \quad \text{for all } t \text{ and } u. \tag{2.33}$$

Equality (2.32) can be rewritten in terms of $f(t,u,x)$ as follows:

$$\frac{dx_0(t,u)}{dt}=A(t,u)x_0(t,u)+f[t,u,x_0(t,u)]+\eta(t,u). \tag{2.34}$$

Since $x_0(t,u)$ is pseudoperiodic in t and u with periods ω_0 and ω, by the results of [2] stated in **1** we have

$$x_0(t,u)=\int_{-\infty}^{\infty} G(t,s,u)\{f[s,u,x_0(s,u)]+\eta(s,u)\}\,ds. \tag{2.35}$$

Now starting from $x_0(t,u)$ we consider the iterative process

$$x_{n+1}(t,u)=\int_{-\infty}^{\infty}G(t,s,u)f[s,u,x_n(s,u)]ds \quad (n=0,1,2,\cdots). \tag{2.36}$$

Then, since

$$x_1(t,u)=\int_{-\infty}^{\infty}G(t,s,u)f[s,u,x_0(s,u)]ds, \tag{2.37}$$

subtracting (2.35) from (2.37) we have

$$x_1(t,u)-x_0(t,u)=-\int_{-\infty}^{\infty}G(t,s,u)\eta(s,u)ds. \tag{2.38}$$

Then by (2.26), (2.27) and (2.33) we have

$$\|x_1-x_0\|_n\leqq Mr. \tag{2.39}$$

For the iterative process (2.36), as for (2.11), we can prove that it can be continued indefinitely and that

$$\|x_{n+1}-x_n\|_n\leqq\kappa^n\|x_1-x_0\|_n \quad (n=0,1,2,\cdots), \tag{2.40}$$

$$\|x_{n+1}-x_0\|_n<\delta \quad (n=0,1,2,\cdots). \tag{2.41}$$

In fact, for $n=0$, (2.40) is evident and (2.41) follows from (2.39) by (iii) of (2.25). Assume that (2.40) and (2.41) are valid up to $n-1$. Then from (2.36) we have

$$x_{n+1}(t,u)-x_n(t,u)=\int_{-\infty}^{\infty}G(t,s,u)\{f[s,u,x_n(s,u)]-f[s,u,x_{n-1}(s,u)]\}ds.$$

However by (2.31) it holds that

$$\begin{aligned}&f[s,u,x_n(s,u)]-f[s,u,x_{n-1}(s,u)]\\&=X[s,u,x_n(s,u)]-X[s,u,x_{n-1}(s,u)]-A(s,u)[x_n(s,u)-x_{n-1}(s,u)]\\&=\int_0^1\{\Psi[s,u,x_{n-1}(s,u)+\theta(x_n(s,u)-x_{n-1}(s,u))]-A(s,u)\}\\&\quad\times[x_n(s,u)-x_{n-1}(s,u)]d\theta.\end{aligned}$$

Thus by (ii) of (2.25) we have

$$\|x_{n+1}-x_n\|_n\leqq M\cdot\frac{\kappa}{M}\cdot\|x_n-x_{n-1}\|_n=\kappa\|x_n-x_{n-1}\|_n, \tag{2.42}$$

from which follows (2.40) for n since

$$\|x_n-x_{n-1}\|_n\leqq\kappa^{n-1}\|x_1-x_0\|_n$$

by the assumption. Then

$$\begin{aligned}\|x_{n+1}-x_0\|_n&\leqq\|x_{n+1}-x_n\|_n+\|x_n-x_{n-1}\|_n+\cdots+\|x_1-x_0\|_n\\&\leqq(\kappa^n+\kappa^{n-1}+\cdots+\kappa+1)\|x_1-x_0\|_n\\&\leqq\frac{1}{1-\kappa}\|x_1-x_0\|_n,\end{aligned}$$

from which by (2.39) and (iii) of (2.25) follows

$$\|x_{n+1}-x_0\|_n \leqq \frac{Mr}{1-\kappa} < \delta. \tag{2.43}$$

Thus we see that the iterative process (2.36) can be continued indefinitely and (2.40) and (2.41) are valid for all n. Then we have an infinite sequence $\{x_n(t, u)\}$, which by (2.40) is uniformly convergent for all t and u. Hence we have a continuous vector-valued function $\hat{x}(t, u)$ such that

$$\hat{x}(t, u) = \lim_{n\to\infty} x_n(t, u). \tag{2.44}$$

In the same way as the proof of Theorem 1, it is easily seen that $\hat{x}(t, u)$ is pseudoperiodic in t and u with periods ω_0 and ω. Moreover letting $n\to\infty$ in (2.43) we have (2.28). Then in the same way as the proof of Theorem 1, we see that $x=\hat{x}(t, u)$ is a solution to (2.30), that is, (2.21). Thus we see that the given system (2.21) has a solution $x=\hat{x}(t, u)$ pseudoperiodic in t and u with periods ω_0 and ω satisfying (2.28).

The uniqueness of pseudoperiodic solutions to (2.21) satisfying (2.29) can be proved in the same way as the proof of Theorem 1. Q.E.D.

3. Continuous pseudoperiodic functions corresponding to quasiperiodic functions.

Let $f(t)$ be a matrix-valued function quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$. Then by our definition stated in **1**, there is a continuous matrix-valued function $f^{(0)}(t, u)=f^{(0)}(t, u_1, u_2, \cdots, u_m)$ periodic in t, $u_1, u_2, \cdots, u_m$ with periods $\omega_0, \omega_1, \omega_2, \cdots, \omega_m$ such that

$$f(t)=f^{(0)}(t, t)=f^{(0)}(t, t, t, \cdots, t). \tag{3.1}$$

We shall show that *we may assume without loss of generality*

$$\sum_{i=0}^{m} \frac{r_i}{\omega_i} \neq 0 \tag{3.2}$$

for any rational numbers $r_0, r_1, \cdots, r_m$ *except* $r_0=r_1=\cdots=r_m=0$. Suppose that (3.2) does not hold for some rational numbers $r_0, r_1, \cdots, r_m$ not all vanishing. Then it is clear that

$$\sum_{i=0}^{m} \frac{p_i}{\omega_i} = 0 \tag{3.3}$$

for some integers $p_0, p_1, \cdots, p_m$ not all vanishing. Without loss of generality we may assume that

$$p_m \neq 0. \tag{3.4}$$

Then from (3.3) we obtain the equality of the form

$$\frac{1}{\omega_m}=\sum_{i=0}^{m-1}\frac{p_i'}{q\omega_i}, \tag{3.5}$$

where $q\neq 0$, $p_0', p_1', \cdots, p_{m-1}'$ are all integers. Put

$$f^{(0)}(\omega_0 t, \omega_1 u_1, \omega_2 u_2, \cdots, \omega_m u_m)=\tilde{f}^{(0)}(t, u_1, u_2, \cdots, u_m), \tag{3.6}$$

then $\tilde{f}^{(0)}(t, u_1, u_2, \cdots, u_m)$ is a continuous matrix-valued function periodic in $t, u_1, u_2, \cdots, u_m$ with period unity. From (3.5) and (3.6) we then have

$$\begin{aligned} f^{(0)}(t, u_1, u_2, \cdots, u_m) &= \tilde{f}^{(0)}\left(\frac{t}{\omega_0}, \frac{u_1}{\omega_1}, \cdots, \frac{u_m}{\omega_m}\right) \\ &= \tilde{f}^{(0)}\left(\frac{t}{\omega_0}, \frac{u_1}{\omega_1}, \cdots, \frac{u_{m-1}}{\omega_{m-1}}, \frac{p_0'}{q\omega_0}u_m+\frac{p_1'}{q\omega_1}u_m+\cdots+\frac{p_{m-1}'}{q\omega_{m-1}}u_m\right). \end{aligned} \tag{3.7}$$

We now consider the function

$$\begin{aligned} & f^{(1)}(t, u_1, u_2, \cdots, u_{m-1}) \\ &= \tilde{f}^{(0)}\left(\frac{t}{\omega_0}, \frac{u_1}{\omega_1}, \cdots, \frac{u_{m-1}}{\omega_{m-1}}, \frac{p_0'}{q\omega_0}t+\frac{p_1'}{q\omega_1}u_1+\cdots+\frac{p_{m-1}'}{q\omega_{m-1}}u_{m-1}\right). \end{aligned} \tag{3.8}$$

Then clearly $f^{(1)}(t, u_1, u_2, \cdots, u_{m-1})$ is a continuous matrix-valued function periodic in $t, u_1, u_2, \cdots, u_{m-1}$ with periods $q\omega_0, q\omega_1, \cdots, q\omega_{m-1}$. Moreover by (3.1), (3.7) and (3.8) we have

$$\begin{aligned} f(t) &= f^{(0)}(t, t, t, \cdots, t) \\ &= f^{(1)}(t, t, \cdots, t). \end{aligned} \tag{3.9}$$

This means that the number of periods of $f(t)$ can be reduced. Hence if $\omega_0, \omega_1, \cdots, \omega_m$ are periods of the least number, then we have (3.2). This shows that without loss of generality we may assume (3.2) for any rational numbers $r_0, r_1, \cdots, r_m$ except $r_0=r_1=\cdots=r_m=0$.

In what follows, this will be assumed always.

Then by well known Kronecker's theorem on Diophantine approximations, for arbitrary t and $u=(u_1, u_2, \cdots, u_m)$, to any number $\delta>0$ correspond a number τ and $m+1$ integers $p_0, p_1, \cdots, p_m$ such that

$$|t+p_0\omega_0-\tau|, \ |u_i+p_i\omega_i-\tau|<\delta \quad (i=1, 2, \cdots, m). \tag{3.10}$$

From this result we have

Lemma 1. *Let $f(t)$ be an arbitrary matrix-valued function quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$. Then a continuous pseudoperiodic function $\bar{f}(t, u)=\bar{f}(t, u_1, u_2, \cdots, u_m)$ corresponding to $f(t)$ is uniquely determined and moreover for arbitrary t and u, to any positive number ε corresponds a number τ such that*

(3.11) $$\|\bar{f}(t, u)-f(\tau)\|<\varepsilon.$$

Proof. By the definition of quasiperiodic functions, there is a continuous matrix-valued function $\bar{f}(t, u)=\bar{f}(t, u_1, u_2, \cdots, u_m)$ pseudoperiodic in t and $u=(u_1, u_2, \cdots, u_m)$ with periods ω_0 and $\omega=(\omega_1, \omega_2, \cdots, \omega_m)$ satisfying the equality

(3.12) $$\bar{f}(t, 0)=f(t).$$

Put

(3.13) $$\bar{f}(t, u-t)=f^{(0)}(t, u)=f^{(0)}(t, u_1, u_2, \cdots, u_m),$$

then by the definition of pseudoperiodic functions $f^{(0)}(t, u_1, u_2, \cdots, u_m)$ is clearly periodic in $t, u_1, u_2, \cdots, u_m$ with periods $\omega_0, \omega_1, \omega_2, \cdots, \omega_m$. Since $\bar{f}(t, u)$ is continuous for all t and u, $f^{(0)}(t, u_1, u_2, \cdots, u_m)$ is also continuous for all $t, u_1, u_2, \cdots, u_m$. Then $f^{(0)}(t, u_1, u_2, \cdots, u_m)$ is uniformly continuous for all $t, u_1, u_2, \cdots, u_m$ Then for any positive number ε there is a positive number δ such that

(3.14) $$\|f^{(0)}(t', u_1', u_2', \cdots, u_m')-f^{(0)}(t, u_1, u_2, \cdots, u_m)\|<\varepsilon$$

whenever

(3.15) $$|t'-t|,\ |u_i'-u_i|<\delta \quad (i=1, 2, \cdots, m).$$

Now from (3.13) it is clear that

(3.16) $$\bar{f}(t, u)=f^{(0)}(t, u+t)=f^{(0)}(t, u_1+t, u_2+t, \cdots, u_m+t).$$

For arbitrary t and u, by Kronecker's theorem mentioned above, there are a number τ and integers $p_0, p_1, p_2, \cdots, p_m$ such that

(3.17) $$|t+p_0\omega_0-\tau|,\ |(u_i+t)+p_i\omega_i-\tau|<\delta \quad (i=1, 2, \cdots, m).$$

Then by (3.14) we have

$$\|f^{(0)}(t+p_0\omega_0, u_1+t+p_1\omega_1, \cdots, u_m+t+p_m\omega_m)-f^{(0)}(\tau, \tau, \cdots, \tau)\|<\varepsilon,$$

which by the periodicity of $f^{(0)}(t, u_1, u_2, \cdots, u_m)$ implies

$$\|f^{(0)}(t, u_1+t, u_2+t, \cdots, u_m+t)-f^{(0)}(\tau, \tau, \cdots, \tau)\|<\varepsilon.$$

By (3.16) this means that

(3.18) $$\|\bar{f}(t, u)-\bar{f}(\tau, 0)\|<\varepsilon,$$

which by (3.12) proves (3.11).

Now suppose that there are two continuous pseudoperiodic functions $\bar{f}'(t, u)$ and $\bar{f}''(t, u)$ corresponding to $f(t)$. Then, as is seen from the proof of (3.18), for any positive number ε there is a number τ such that

$$\|\bar{f}'(t, u)-f(\tau)\|,\ \|\bar{f}''(t, u)-f(\tau)\|<\varepsilon.$$

Then we have

$$\|\bar{f}'(t, u) - \bar{f}''(t, u)\| < 2\varepsilon.$$

Since ε is an arbitrary positive number, this implies

$$\|\bar{f}'(t, u) - \bar{f}''(t, u)\| = 0,$$

that is,

$$\bar{f}'(t, u) = \bar{f}''(t, u).$$

Since t and u are arbitrary, this means that

$$\bar{f}'(t, u) = \bar{f}''(t, u) \qquad \text{for all } t \text{ and } u.$$

This proves the uniqueness of continuous pseudoperiodic functions corresponding to $f(t)$. Q.E.D.

From the proof of Lemma 1, we readily see that the following lemma is also valid.

Lemma 2. *Let $f_i(t)$ $(i=1, 2, \cdots, n)$ be arbitrary matrix-valued functions quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ and*

$$\bar{f}_i(t, u) = \bar{f}_i(t, u_1, u_2, \cdots, u_m) \quad (i=1, 2, \cdots, n)$$

be continuous pseudoperiodic functions corresponding to $f_i(t)$ respectively. Then for arbitrary t and u, to any positive number ε corresponds a number τ such that

$$\|\bar{f}_i(t, u) - f_i(\tau)\| < \varepsilon \quad (i=1, 2, \cdots, n).$$

4. Quasiperiodic solutions to nonlinear quasiperiodic differential systems.

On the basis of Theorem 1 we have

Theorem 3. *Given a quasiperiodic differential system of the form*

$$\frac{dx}{dt} = A(t)x + f(t, x), \tag{4.1}$$

where $A(t)$ is a matrix quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ and $f(t, x)$ is a continuous vector-valued function quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ for all $x \in D$. Here D is a closed region of the x-space.

Suppose that the quasiperiodic differential operator L defined by

$$Ly = \frac{dy}{dt} - A(t)y \tag{4.2}$$

is regular and that the Green function $\bar{G}(t, s, u)$ of the pseudoperiodic differential operator $\bar{L}$ corresponding to L satisfies

$$\|\bar{G}(t, s, u)\| \leqq K e^{-\gamma |t-s|} \qquad \text{for all } t, s \text{ and } u, \tag{4.3}$$

where K and γ are positive numbers. Assume that

(4.4) $$\|f(t,x')-f(t,x'')\|\leqq\frac{\kappa}{M}\|x'-x''\|$$

for all t and any $x', x''\in D$, where

(4.5) $$M=\frac{2K}{\gamma}$$

and κ is a non-negative number smaller than unity.

If there is a vector-valued function $x_0(t)$ quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ such that

(4.6) $$x_0(t)\in D \qquad \text{for all } t$$

and

(4.7) $$S=\left\{x \,|\, \|x-x_1(t)\|\leqq\frac{\kappa}{1-\kappa}\|x_1-x_0\|_n \quad \text{for some } t\right\}\subset D,$$

then the given system (4.1) has a unique solution $x=\hat{x}(t)$ quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ such that

(4.8) $$\hat{x}(t)\in D \qquad \text{for all } t,$$

and moreover for $\hat{x}(t)$ it holds that

(4.9) $$\|\hat{x}(t)-x_0(t)\|\leqq\frac{1}{1-\kappa}\|x_1-x_0\|_n.$$

In (4.7), however,

(4.10) $$x_1(t)=\int_{-\infty}^{\infty}G(t,s)f[s,x_0(s)]ds$$

where $G(t,s)$ is the Green function of the quasiperiodic differential operator L, that is,

(4.11) $$G(t,s)=\bar{G}(t,s,0),$$

and $\|f\|_n$ denotes $\sup_t\|f(t)\|$ for any vecotr-valued function $f(t)$ bounded for all t.

Proof. Let $\bar{f}(t,u,x)$ be a vector-valued function continuous and pseudoperiodic in t and u with periods ω_0 and $\omega=(\omega_1, \omega_2, \cdots, \omega_m)$ such that

(4.12) $$\bar{f}(t,0,x)=f(t,x).$$

Let x' and x'' be arbitrary points belonging to D. Then by Lemma 2, for arbitrary t and u, to any positive number ε corresponds a number τ such that

$$\|\bar{f}(t,u,x')-f(\tau,x')\|<\frac{\varepsilon}{2},$$

$$\|\bar{f}(t,u,x'')-f(\tau,x'')\|<\frac{\varepsilon}{2}.$$

Then by (4.4) we have

$$\|\bar{f}(t,u,x')-\bar{f}(t,u,x'')\|<\varepsilon+\frac{\kappa}{M}\|x'-x''\|.$$

Since ε is an arbitrary positive number, letting $\varepsilon\to 0$, we have

$$\|\bar{f}(t,u,x')-\bar{f}(t,u,x'')\|\leqq\frac{\kappa}{M}\|x'-x''\|. \tag{4.13}$$

Since t and u are arbitrary and x', x'' are arbitrary points belonging to D, (4.13) is valid for all t and u and any $x', x''\in D$. From (4.13) it readily follows that $\bar{f}(t,u,x)$ is continuous in t, u and x.

Now let $\bar{x}_0(t,u)$ be the continuous pseudoperiodic vector-valued function corresponding to $x_0(t)$. Then by Lemma 1, for arbitrary t and u, to any positive number ε corresponds a number τ such that

$$\|\bar{x}_0(t,u)-x_0(\tau)\|<\varepsilon. \tag{4.14}$$

Since $x_0(t)\in D$ for all t, $\bar{x}_0(t,u)$ then belongs to the ε-neighborhood of D. Since ε is an arbitrary positive number, this implies

$$\bar{x}_0(t,u)\in\bar{D}, \tag{4.15}$$

where $\bar{D}$ is the closure of D. Since D is closed by the assumption and t and u are arbitrary, (4.15) implies

$$\bar{x}_0(t,u)\in D \quad \text{for all } t \text{ and } u. \tag{4.16}$$

Then put

$$\bar{x}_1(t,u)=\int_{-\infty}^{\infty}\bar{G}(t,s,u)\bar{f}[s,u,\bar{x}_0(s,u)]ds, \tag{4.17}$$

then by (4.11), (4.12) and (4.10) it is clear that

$$\bar{x}_1(t,0)=\int_{-\infty}^{\infty}G(t,s)f[s,x_0(s)]ds=x_1(t). \tag{4.18}$$

Since $\bar{x}_1(t,u)$ is continuous and is pseudoperiodic in t and u with periods ω_0 and ω, it is a continuous pseudoperiodic vector-valued function corresponding to $x_1(t)$.

Now consider x such that

$$\|x-\bar{x}_1(t,u)\|\leqq\frac{\kappa}{1-\kappa}\|\bar{x}_1-\bar{x}_0\|_n. \tag{4.19}$$

First we shall prove that

$$\|\bar{x}_1-\bar{x}_0\|_n=\|x_1-x_0\|_n. \tag{4.20}$$

By Lemma 2, for arbitrary t and u, to any positive number ε corresponds a number τ such that

$$\|\bar{x}_1(t,u)-x_1(\tau)\|,\ \|\bar{x}_0(t,u)-x_0(\tau)\|<\frac{\varepsilon}{2}.$$

Then we have

$$\|\bar{x}_1(t,u)-\bar{x}_0(t,u)\|<\varepsilon+\|x_1(\tau)-x_0(\tau)\|\leqq\varepsilon+\|x_1-x_0\|_n.$$

Since ε is an arbitrary positive number, letting $\varepsilon\to 0$ we have

$$\|\bar{x}_1(t,u)-\bar{x}_0(t,u)\|\leqq\|x_1-x_0\|_n.$$

Since t and u are arbitrary, we thus have

$$\|\bar{x}_1-\bar{x}_0\|_n=\sup_{t,u}\|\bar{x}_1(t,u)-\bar{x}_0(t,u)\|\leqq\|x_1-x_0\|_n.$$

On the other hand,

$$\|x_1(t)-x_0(t)\|=\|\bar{x}_1(t,0)-\bar{x}_0(t,0)\|\leqq\|\bar{x}_1-\bar{x}_0\|_n$$

for any t. Hence clearly

$$\|x_1-x_0\|_n\leqq\|\bar{x}_1-\bar{x}_0\|_n.$$

Thus we have (4.20).

Now for arbitrary t and u, to any positive number ε corresponds a number τ such that

$$\|\bar{x}_1(t,u)-x_1(\tau)\|<\varepsilon.$$

Then for x satisfying (4.19) we have

$$\begin{aligned}\|x-x_1(\tau)\|&<\frac{\kappa}{1-\kappa}\|\bar{x}_1-\bar{x}_0\|_n+\varepsilon\\&=\frac{\kappa}{1-\kappa}\|x_1-x_0\|_n+\varepsilon.\end{aligned}$$

This implies that x belongs to the ε-neighborhood of S. Since ε is an arbitrary positive number, we then see that

$$x\in\bar{D}=D,$$

that is,

$$\left\{x \mid \|x-\bar{x}_1(t,u)\|\leqq\frac{\kappa}{1-\kappa}\|\bar{x}_1-\bar{x}_0\|_n \text{ for some } t \text{ and } u\right\}\subset D. \tag{4.21}$$

Let $\bar{A}(t,u)$ be a continuous pseudoperiodic matrix corresponding to $A(t)$. Then by (4.13) and (4.21) we see that the conditions of Theorem 1 are all fulfilled for the pseudoperiodic differential system

$$\frac{dx}{dt}=\bar{A}(t,u)x+\bar{f}(t,u,x). \tag{4.22}$$

Hence by Theorem 1 we see that (4.22) has a unique solution $x=\hat{x}(t,u)$ pseudo-periodic in t and u with periods ω_0 and ω such that

$$\hat{x}(t,u)\in D \quad \text{for all } t \text{ and } u \tag{4.23}$$

and

$$\|\hat{x}(t,u)-\bar{x}_0(t,u)\|\leqq\frac{1}{1-\kappa}\|\bar{x}_1-\bar{x}_0\|_n. \tag{4.24}$$

Put

$$\hat{x}(t,0)=\hat{x}(t), \tag{4.25}$$

then clearly $x=\hat{x}(t)$ is a solution to (4.1) quasiperiodic in t with periods ω_0, $\omega_1,\cdots,\omega_m$ and by (4.24) it satisfies the inequality (4.9).

It now remains to prove the uniqueness of quasiperiodic solutions. Let $x=\hat{x}(t)$ be an arbitrary solution to (4.1) quasiperiodic in t with periods $\omega_0,\omega_1,\cdots,\omega_m$ such that

$$\hat{x}(t)\in D \quad \text{for all } t.$$

Then clearly $x=\hat{x}(t)$ satisfies the equation

$$\frac{dx}{dt}=\bar{A}(t,0)x+\bar{f}(t,0,x).$$

Since $\hat{x}(t)$ is bounded for all t, by the result of [2] stated in **1**, we have

$$\begin{aligned}\hat{x}(t)&=\int_{-\infty}^{\infty}\bar{G}(t,s,0)\bar{f}[s,0,\hat{x}(s)]ds\\&=\int_{-\infty}^{\infty}G(t,s)f[s,\hat{x}(s)]ds.\end{aligned} \tag{4.26}$$

Suppose that (4.1) has two solutions $\hat{x}(t)$ and $\hat{x}'(t)$ quasiperiodic in t with periods $\omega_0,\omega_1,\cdots,\omega_m$ such that

$$\hat{x}(t),\hat{x}'(t)\in D \quad \text{for all } t.$$

Then besides (4.26) we have

$$\hat{x}'(t)=\int_{-\infty}^{\infty}G(t,s)f[s,\hat{x}'(s)]ds. \tag{4.27}$$

Then subtracting (4.26) from (4.27), by (4.3), (4.4) and (4.5) we have

$$\|\hat{x}'(t)-\hat{x}(t)\|\leqq\kappa\|\hat{x}'-\hat{x}\|_n,$$

which implies

$$\|\hat{x}'-\hat{x}\|_n \leqq \kappa \|\hat{x}'-\hat{x}\|_n.$$

Since $0 \leqq \kappa < 1$, this implies

$$\|\hat{x}'-\hat{x}\|_n = 0,$$

that is,

$$\hat{x}'(t) \equiv \hat{x}(t).$$

This proves the uniqueness of quasiperiodic solutions to (4.1). Q.E.D.

On the basis of Theorem 2 we have

Theorem 4. *Given a nonlinear quasiperiodic differential system*

$$\frac{dx}{dt} = X(t, x), \tag{4.28}$$

where $X(t,x)$ is quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ and the pseudoperiodic function $\bar{X}(t,u,x)$ corresponding to $X(t,x)$ is continuously differentiable with respect to x for $x \in D$. Here D is a closed region of the x-space.

Suppose that the differential system (4.28) has an approximate solution $x = x_0(t)$ quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ such that the continuous pseudoperiodic function $\bar{x}_0(t,u)$ corresponding to $x_0(t)$ is continuously differentiable with respect to t for all t and u, and

$$x_0(t) \in D \quad \text{for all } t, \tag{4.29}$$

$$\left\| \frac{dx_0(t)}{dt} - X[t, x_0(t)] \right\| \leqq r \quad \text{for all } t. \tag{4.30}$$

Further suppose that there are a positive number δ, a non-negative number $\kappa < 1$ and a matrix $A(t)$ quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ satisfying the conditions as follows:

1° *the quasiperiodic differential operator L defined by*

$$Ly = \frac{dy}{dt} - A(t)y \tag{4.31}$$

is regular,

2°

$$\begin{cases} \text{(i)} \quad D_\delta = \{x \mid \|x - x_0(t)\| < \delta \ \text{ for some } t\} \subset D, \\ \text{(ii)} \quad \|\Psi(t,x) - A(t)\| \leqq \dfrac{\kappa}{M} \ \text{ for any } (t,x) \text{ satisfying } \|x - x_0(t)\| < \delta, \\ \text{(iii)} \quad \dfrac{Mr}{1-\kappa} < \delta. \end{cases} \tag{4.32}$$

Here $\Psi(t,x)$ is the Jacobian matrix of $X(t,x)$ with respect to x and

$$M = \frac{2K}{\gamma}, \tag{4.33}$$

where K and γ are positive numbers such that the Green function $\bar{G}(t,s,u)$ of the pseudoperiodic differential operator corresponding to L satisfies

(4.34) $$\|\bar{G}(t,s,u)\| \leqq Ke^{-\gamma|t-s|} \quad \textit{for all } t, s \textit{ and } u.$$

The given differential system (4.28) then has a solution $x=\hat{x}(t)$ quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ such that

(4.35) $$\|\hat{x}-x_0\|_n \leqq \frac{Mr}{1-\kappa} < \delta.$$

Moreover a solution $x=\hat{x}(t)$ to (4.28) quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ satisfying the inequality

(4.36) $$\|\hat{x}(t)-x_0(t)\| < \delta \quad \textit{for all } t$$

is unique.

Proof. As is shown in the proof of Theorem 3,

(4.37) $$\bar{x}_0(t,u) \in D \quad \text{for all } t \text{ and } u.$$

Consider

(4.38) $$\frac{d\bar{x}_0(t,u)}{dt} - \bar{X}[t,u,\bar{x}_0(t,u)].$$

This is continuous and is pseudoperiodic in t and u with periods ω_0 and $\omega=(\omega_1, \omega_2, \cdots, \omega_m)$. Therefore by Lemma 1, for arbitrary t and u, to any positive number ε corresponds a number τ such that

$$\left\|\left[\frac{d\bar{x}_0(t,u)}{dt} - \bar{X}[t,u,\bar{x}_0(t,u)]\right] - \left[\frac{dx_0(t)}{dt} - X[t,x_0(t)]\right]_{t=\tau}\right\| < \varepsilon.$$

Then by the assumption (4.30) it follows that

$$\left\|\frac{d\bar{x}_0(t,u)}{dt} - \bar{X}[t,u,\bar{x}_0(t,u)]\right\| < r+\varepsilon.$$

Since ε is an arbitrary positive number, letting $\varepsilon \to 0$ we have

$$\left\|\frac{d\bar{x}_0(t,u)}{dt} - \bar{X}[t,u,\bar{x}_0(t,u)]\right\| \leqq r.$$

Since t and u are arbitrary, this implies

(4.39) $$\left\|\frac{d\bar{x}_0(t,u)}{dt} - \bar{X}[t,u,\bar{x}_0(t,u)]\right\| \leqq r \quad \text{for all } t \text{ and } u.$$

Now consider x such that

(4.40) $$\|x-\bar{x}_0(t,u)\| < \delta.$$

Then there is a positive number $\delta_1 < \delta$ such that

$$\|x-\bar{x}_0(t,u)\|\leqq\delta_1<\delta. \tag{4.41}$$

Let ε be an arbitrary positive number such that

$$0<\varepsilon<\delta-\delta_1. \tag{4.42}$$

Since $\bar{x}_0(t,u)$ is continuous and is pseudoperiodic in t and u with periods ω_0 and ω, by Lemma 1 there is a number τ such that

$$\|\bar{x}_0(t,u)-x_0(\tau)\|<\varepsilon.$$

Then from (4.41) and (4.42) we have

$$\|x-x_0(\tau)\|<\delta.$$

By (i) of (4.32), we then see that $x\in D_\delta\subset D$. This proves that

$$\bar{D}_\delta=\{x|\ \|x-\bar{x}_0(t,u)\|<\delta \quad \text{for some } t \text{ and } u\}\subset D. \tag{4.43}$$

Next for x satisfying (4.40), consider

$$\|\bar{\Psi}(t,u,x)-\bar{A}(t,u)\|,$$

where $\bar{\Psi}(t,u,x)$ is the Jacobian matrix of $\bar{X}(t,u,x)$ with respect to x and $\bar{A}(t,u)$ is the continuous pseudoperiodic matrix corresponding to $A(t)$. Since $\bar{\Psi}(t,u,x)-\bar{A}(t,u)$ is continuous and is pseudoperiodic in t and u with periods ω_0 and ω for any x fixed, by Lemma 2 for any positive number ε satisfying (4.42) there is a number τ such that

$$\|\bar{x}_0(t,u)-x_0(\tau)\|<\varepsilon,$$
$$\|[\bar{\Psi}(t,u,x)-\bar{A}(t,u)]-[\Psi(\tau,x)-A(\tau)]\|<\varepsilon.$$

Then for any x satisfying (4.41) we have

$$\|x-x_0(\tau)\|<\delta.$$

Therefore by (ii) of (4.32) we have

$$\|\bar{\Psi}(t,u,x)-\bar{A}(t,u)\|<\frac{\kappa}{M}+\varepsilon.$$

Since ε is an arbitrary positive number satisfying (4.42), letting $\varepsilon\to 0$ we have

$$\|\bar{\Psi}(t,u,x)-\bar{A}(t,u)\|\leqq\frac{\kappa}{M}. \tag{4.44}$$

This proves that (4.44) is valid for x satisfying (4.40).

By (4.37), (4.39), (4.43) and (4.44), we see that all the conditions of Theorem 2 are fulfilled for the pseudoperiodic differential system

$$\frac{dx}{dt}=\bar{X}(t,u,x). \tag{4.45}$$

Hence by Theorem 2 we see that (4.45) has a solution $x=\tilde{x}(t,u)$ pseudoperiodic in t and u with periods ω_0 and ω such that

(4.46) $$\|\tilde{x}-\tilde{x}_0\|_n \leqq \frac{Mr}{1-\kappa} < \delta.$$

Put

$$\tilde{x}(t,0)=\hat{x}(t),$$

then clearly $x=\hat{x}(t)$ is a solution to (4.28) quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ and by (4.46) it satisfies the inequality (4.35).

It now remains to prove the uniqueness of quasiperiodic solution. Let $x=\hat{x}(t)$ be an arbitrary solution to (4.28) quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ such that

$$\|\hat{x}(t)-x_0(t)\|<\delta \quad \text{for all } t.$$

Then

(4.47) $$\frac{d\hat{x}(t)}{dt}=X[t,\hat{x}(t)]=A(t)\hat{x}(t)+f[t,\hat{x}(t)],$$

where

(4.48) $$f(t,x)=X(t,x)-A(t)x.$$

Since $\hat{x}(t)$ is bounded for all t, similarly to (4.26) we have

(4.49) $$\hat{x}(t)=\int_{-\infty}^{\infty} G(t,s)f[s,\hat{x}(s)]ds.$$

Suppose that (4.28) has two solutions $\hat{x}(t)$ and $\hat{x}'(t)$ quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ such that

(4.50) $$\|\hat{x}(t)-x_0(t)\|,\ \|\hat{x}'(t)-x_0(t)\|<\delta \qquad \text{for all } t.$$

Then besides (4.49) we have

(4.51) $$\hat{x}'(t)=\int_{-\infty}^{\infty} G(t,s)f[s,\hat{x}'(s)]ds.$$

Now by (4.48)

$$\begin{aligned} & f[s,\hat{x}'(s)]-f[s,\hat{x}(s)] \\ &=\{X[s,\hat{x}'(s)]-X[s,\hat{x}(s)]\}-A(s)[\hat{x}'(s)-\hat{x}(s)] \\ &=\int_0^1 \{\Psi[s,\hat{x}+\theta(\hat{x}'-\hat{x})]-A(s)\}[\hat{x}'(s)-\hat{x}(s)]d\theta. \end{aligned}$$

Therefore on account of (4.50), by (ii) of (4.32) we have

$$\|f[s,\hat{x}'(s)]-f[s,\hat{x}(s)]\| \leqq \frac{\kappa}{M}\|\hat{x}'(s)-\hat{x}(s)\|.$$

Then subtracting (4.49) from (4.51), by (4.33) and (4.34) we have

$$\|\hat{x}'-\hat{x}\|_n \leqq \kappa \|\hat{x}'-\hat{x}\|_n.$$

Since $0 \leqq \kappa < 1$, this implies

$$\|\hat{x}'-\hat{x}\|_n = 0,$$

that is,

$$\hat{x}'(t) \equiv \hat{x}(t).$$

This proves the uniqueness of quasiperiodic solutions to (4.28). Q.E.D.

From Theorem 4, we can get the following theorem concerned with the perturbation of quasiperiodic differential systems.

Theorem 5. *Given a nonlinear quasiperiodic differential system of the form*

$$\frac{dx}{dt} = X(t, x) + \varepsilon F(t, x, \varepsilon), \tag{4.52}$$

where $X(t, x)$ *and* $F(t, x, \varepsilon)$ *are both quasiperiodic in* t *with periods* $\omega_0, \omega_1, \cdots, \omega_m$ *for any* $x \in D$ *and any* ε *satisfying*

$$|\varepsilon| \leqq \varepsilon_0 \quad (\varepsilon_0 > 0). \tag{4.53}$$

Here D *is a closed bounded region of the* x*-space.*

Suppose that the continuous pseudoperiodic functions $\bar{X}(t, u, x)$ *and* $\bar{F}(t, u, x, \varepsilon)$ *corresponding to* $X(t, x)$ *and* $F(t, x, \varepsilon)$ *respectively are both continuously differentiable with respect to* x *for all* t *and* u, $x \in D$ *and* ε *satisfying* (4.53), *and that the Jacobian matrix* $\Psi(t, x)$ *of* $X(t, x)$ *with respect to* x *satisfies a Lipschitz condition* :

$$\|\Psi(t, x') - \Psi(t, x'')\| \leqq L \|x' - x''\| \quad (L > 0) \tag{4.54}$$

for all t *and any* $x', x'' \in D$.

Assume that the unperturbed system of (4.52), *that is, the system*

$$\frac{dx}{dt} = X(t, x) \tag{4.55}$$

has a solution $x = x_0(t)$ *quasiperiodic in* t *with periods* $\omega_0, \omega_1, \cdots, \omega_m$ *such that the pseudoperiodic function* $\bar{x}_0(t, u)$ *corresponding to* $x_0(t)$ *is continuously differentiable with respect to* t *for all* t *and* u, *and*

$$x_0(t) \in D \quad \text{for all } t, \tag{4.56}$$

$$D_0 = \{x \mid \|x - x_0(t)\| < \delta_0 \ \text{ for some } t\} \subset D \quad \text{for some } \delta_0 > 0, \tag{4.57}$$

and the quasiperiodic differential operator L *defined by*

$$Ly = \frac{dy}{dt} - \Psi[t, x_0(t)]y \tag{4.58}$$

is regular.

Then there is a positive number $\varepsilon_1 \leqq \varepsilon_0$ *such that for any* ε *satisfying*

$$|\varepsilon| < \varepsilon_1, \tag{4.59}$$

the given system (4.52) *has a solution* $x = \hat{x}(t, \varepsilon)$ *quasiperiodic in* t *with periods* $\omega_0, \omega_1, \cdots, \omega_m$ *such that*

$$\|\hat{x}(t, \varepsilon) - x_0(t)\| = O(|\varepsilon|) \quad (\varepsilon \to 0). \tag{4.60}$$

Moreover, for any ε *satisfying* (4.59), *a solution* $x = \hat{x}(t, \varepsilon)$ *to* (4.52) *quasiperiodic in* t *with periods* $\omega_0, \omega_1, \cdots, \omega_m$ *is determined uniquely in a certain neighborhood of* $x_0(t)$.

Proof. By the assumption there is a positive number C_0 such that

$$\|F[t, x_0(t), \varepsilon]\| \leqq C_0 \tag{4.61}$$

for all t and ε satisfying (4.53). Then we have

$$\begin{aligned} \left\| \frac{dx_0(t)}{dt} - \{X[t, x_0(t)] + \varepsilon F[t, x_0(t), \varepsilon]\} \right\| & \\ = |\varepsilon| \cdot \|F[t, x_0(t), \varepsilon]\| & \\ \leqq C_0 |\varepsilon|. & \end{aligned} \tag{4.62}$$

Let $F_x(t, x, \varepsilon)$ be the Jacobian matrix of $F(t, x, \varepsilon)$ with respect to x, then by the assumption there is a positive number C such that

$$\|F_x(t, x, \varepsilon)\| \leqq C \tag{4.63}$$

for all $t, x \in D$ and ε satisfying (4.53). Then for any x satisfying

$$\|x - x_0(t)\| < \delta \leqq \delta_0, \tag{4.64}$$

by (4.54) and (4.63) we have

$$\begin{aligned} \|\{\Psi(t, x) + \varepsilon F_x(t, x, \varepsilon)\} - \Psi[t, x_0(t)]\| & \\ \leqq L\|x - x_0(t)\| + C|\varepsilon| & \\ < L\delta + C|\varepsilon|. & \end{aligned} \tag{4.65}$$

Thus we see that the condition (4.32) in Theorem 4 is fulfilled if

$$\frac{MC_0}{1-\kappa} |\varepsilon| < \delta, \tag{4.66}$$

$$L\delta + C|\varepsilon| \leqq \frac{\kappa}{M} \tag{4.67}$$

for some positive number $\delta \leqq \delta_0$ and a non-negative number $\kappa < 1$. Here M is the number connected with the Green function $\bar{G}(t, s, u)$ of the pseudoperiodic differential operator corresponding to L. That is, let $\bar{G}(t, s, u)$ satisfy

$$\|\bar{G}(t, s, u)\| \leqq Ke^{-\gamma|t-s|} \quad \text{for all } t, s \text{ and } u,$$

where K and γ are positive numbers. Then

$$M=\frac{2K}{\gamma}$$

Now (4.66) and (4.67) can be rewritten as follows:

$$\frac{MC_0}{1-\kappa}|\varepsilon|<\delta\leqq\frac{1}{L}\left(\frac{\kappa}{M}-C|\varepsilon|\right), \tag{4.68}$$

which implies

$$\left(\frac{MC_0}{1-\kappa}+\frac{C}{L}\right)|\varepsilon|<\frac{\kappa}{LM},$$

that is,

$$|\varepsilon|<\frac{\kappa}{M}\left(C+\frac{MLC_0}{1-\kappa}\right)^{-1}. \tag{4.69}$$

Let us take κ such that

$$\kappa=\min(\kappa_0, LM\delta_0) \tag{4.70}$$

where κ_0 is an arbitrary positive number smaller than unity. For κ given by (4.70) put

$$\varepsilon_1=\min\left[\varepsilon_0,\ \frac{\kappa}{M}\left(C+\frac{MLC_0}{1-\kappa}\right)^{-1}\right]. \tag{4.71}$$

Then for any ε satisfying

$$|\varepsilon|<\varepsilon_1, \tag{4.72}$$

we have

$$\frac{MC_0}{1-\kappa}|\varepsilon|<\frac{1}{L}\left(\frac{\kappa}{M}-C|\varepsilon|\right)\leqq\frac{\kappa}{LM}\leqq\delta_0.$$

Hence if we take δ so that

$$\delta=\frac{1}{L}\left(\frac{\kappa}{M}-C|\varepsilon|\right), \tag{4.73}$$

then we have (4.68) and $\delta\leqq\delta_0$, in other words, we have a positive number $\delta\leqq\delta_0$ and a positive number $\kappa<1$ satisfying (4.66) and (4.67).

By Theorem 4, we then see that for any ε satisfying (4.72) the given system (4.52) has a solution $x=\hat{x}(t,\varepsilon)$ quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$, which satisfies

$$\|\hat{x}(t,\varepsilon)-x_0(t)\|\leqq\frac{MC_0}{1-\kappa}|\varepsilon|. \tag{4.74}$$

This proves the first conclusion of the theorem.

By Theorem 4, we further see that a solution $x=\hat{x}(t, \varepsilon)$ to (4.52) quasiperiodic in t with periods $\omega_0, \omega_1, \cdots, \omega_m$ satisfying the inequality

$$\|\hat{x}(t, \varepsilon)-x_0(t)\|<\delta=\frac{1}{L}\left(\frac{\kappa}{M}-C|\varepsilon|\right) \tag{4.75}$$

is determined uniquely. This proves the second conclusion of the theorem.

Q.E.D.

Remark. As is seen from the above proof, in Theorem 5, a bound ε_1 of ε within which the existence of a quasiperiodic solution to (4.52) is guaranteed can be given explicitly, say, as (4.71) and a neighborhood of $x_0(t)$ where the uniqueness of such a quasiperiodic solution is guaranteed also can be given explicitly, say, as (4.75).

References

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