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Invariant Positive Measures for Flows in the Plane*

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Introduction.

A measure μ on a topological space X is said to be *invariant with respect to a dynamical system* or *flow* π on X if for every measurable set $A\subset X$ and for every $t\in R$ we have

$$\mu(\pi(A, t))=\mu(A).$$

Since the celebrated recurrence theorem of Poincaré appeared, many important properties of a dynamical system admitting an invariant measure were obtained and a new branch of mathematics called "Ergodic Theory" was developed. For future reference, we recall here the following:

(*) Let π be a dynamical system on a locally compact metric space X with a countable base, and let μ be an invariant measure having the following properties: $\mu(K)$ is finite for every compact set $K\subset X$, and $\mu(X)=\infty$. Then for almost all $x\in X$, the motion $\pi(x, t)$ is either Poisson stable or departing ([3]).

However, very little work has been done on the converse problem, i.e., the problem to establish necessary and sufficient, or sufficient conditions for a given dynamical system to admit an invariant measure. As far as we know, the following are among the most elegant results.

(1) Every dynamical system on a compact metric space admits an invariant measure ([3]).

(2) A dynamical system π on a complete separable metric space X admits a finite invariant measure if and only if there exist a compact set $K\subset X$ and a point $x\in X$ such that

$$\overline{\lim_{\tau\to\infty}}\frac{1}{\tau}\int_0^\tau \phi_K(\pi(x, t))dt>0,$$

where ϕ_K denotes the characteristic function of K ([5]).

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* The paper contains proofs and detailed discussions and explanations of the results published in Proceedings of Japan-U.S. Seminar on Ordinary Differential and Functional Equations, Lectures Notes in Mathematics, Vol. 243, 266-269, Springer Verlag, Berlin Heidelberg New York, 1971.

The measures considered in the results recalled above are not necessarily positive on each non-empty open subset. Motivated by practical applications, the author is interested only in invariant Lebesgue measures satisfying the following two conditions:

(1) *The measure of every non-empty open subset is positive*;

(2) *The measure of every compact subset is finite.*

In the sequel such a measure will be called an *invariant positive measure.*

The main purpose of this paper is to obtain necessary and sufficient conditions for existence of invariant positive measures for local systems on R^2, S^2 or on an open subset of R^2 or S^2. (A similar study for a dynamical system on the torus was made by J. C. Oxtoby. He gave an example of Stepanov flows on the torus which admit an invariant positive measure ([4]). However, it is still unknown, as far as the author knows, whether every Stepanov flow on the torus admits such a measure.)

Our main results are as follows (cf. (*) recalled above):

(I) *Let π be a local system on R^2 or S^2 or on an open subset of R^2 or S^2.*

(i) *If x is a regular point, then π admits an invariant positive measure locally at x.*

(ii) *Let x be an isolated singular point. If x is either a Poincaré center (abbreviated as a center) or a generalized saddle, then π admits an invariant positive measure locally at x.* (For the definition of a measure invariant locally at a point, see §4 below. The necessity was proved in [7]**).

(II) (i) *Let π be a local system on R^2 for which there are only a finite number of singular points. Then π admits an invariant positive measure if and only if the number of non-periodic orbits which have a non-empty limit set (+ or −) is finite.*

(ii) *Let π be a dynamical system on R^2 or S^2 for which there are only a finite number of singular points. Then π admits a finite invariant positive measure if and only if the number of non-periodic orbits is finite.*

In Section 1, we recall the notion of a local system on a topological space and explain basic concepts and notations which will be used in later sections. In Section 2, we recall the notion of an isomorphism between local systems, extend the definition of an invariant measure to local systems, show that our isomorphisms preserve invariant positive measures and discuss existence of invariant positive measures for some special flows. In Section 3, we discuss existence of invariant positive measures for some special flows in R^2, which will

** In [7], the authors consider only dynamical systems defined by autonomous systems of differential equations. However, it is easily seen that their results can be extended to general dynamical systems on any open subset of R^2 or S^2.

be a foundation for the later sections. In Section 4, we introduce the notion of the local existence of an invariant positive measure and discuss the existence for some cases. In Sections 5 and 6, we obtain necessary and sufficient conditions for the global existence of invariant positive measures on R^2 and S^2, respectively, as described above, and thereby achieve the main purpose of our papar.

§1. Basic Concepts.

Throughout the paper, R denotes the set of real numbers.

Definition 1.1. (cf. [1], [8], [9]). Let X be a topological space, let $\mathcal{D}$ be an open subset in $X \times R$ of the form

$$\mathcal{D} = \bigcup_{x \in X} \{x\} \times I_x,$$

where $I_x (=(a_x, b_x))$ is an open interval containing 0 and let π be a mapping of $\mathcal{D}$ into X. We say that π is a *local dynamical system on the phase space* X (or, more briefly, a *local system* or a *local flow on* X) if the following conditions (1)∼(4) are satisfied:

(1) *Identity*: $\pi(x, 0) = x$ for all $x \in X$.

(2) *Continuity*: π is continuous.

(3) *Homomorphism*: If $(x, t) \in \mathcal{D}$, $(x, t+s) \in \mathcal{D}$ and $(\pi(x, t), s) \in \mathcal{D}$, then

$$\pi(\pi(x, t), s) = \pi(x, t+s).$$

(4) *Nonextendability*: If a_x (resp. b_x) is finite, then the cluster set $L^-(x)$, $(L^+(x))$ of $\pi(x, t)$ as $t \downarrow a_x$ $(t \uparrow b_x)$ is empty.

If in particular $I_x = R$ for all $x \in X$, i.e., $\mathcal{D} = X \times R$, we say that π is a *global system*.

$C^+(x)$, $C^-(x)$ and $C(x)$ denote the ***positive semi-orbit***, the ***negative semi-orbit*** and the ***orbit through a point*** x, respectively, e.g. $C(x) = \pi(x, I_x)$. Let $x \in X$ be a periodic point. $T(x)$ denotes the *fundamental* (or prime) period of x. $J^+(x)$ denotes the positive *prolongational limit set* (for the definition of $J^+(x)$ for a local system, see [1], [9]). If $x \notin J^+(x)$ for all $x \in X$, π is said to be *completely unstable*. One may notice that $x \notin J^+(x)$ if and only if there exist an open neighborhood U of x and a positive number $T (< b_x)$ such that

$$\pi(U, t) \cap U = \phi \qquad \text{for all } t > T$$

(If $(U, t) \not\subset \mathcal{D}$, we understand by $\pi(U, t)$ the set $\pi((U, t) \cap \mathcal{D})$. We can replace $t > T$ by $|t| > T$ $(< \min\{|a_x|, b_x\})$, if the phase space is Hausdorff, cf. [1]).

Let $U \subset X$ be an open subset. $\pi \| U$ denotes the restricted local system of π to U (for the definition of $\pi \| U$, see [8]). $M \subset X$ is said to be *positively* (or *negatively*) *quasi-invariant* if $C^+(M)\subset M$ (or $C^-(M)\subset M$).

Let $x\in X$ be an isolated singular point. x is called a *center* if there exists an open neighborhood U of x such that all points of $U-\{x\}$ are periodic. x is called a *generalized saddle* if only a finite number (non-zero) of orbits approach x as $t\uparrow+\infty$ or $t\downarrow-\infty$ (cf. [3], [6]).

Definition 1.2. (cf. [1]). Let π be a local system on X. π is called a *local parallel flow* if the following conditions (1) and (2) are satisfied:

(1) $$X=\bigcup_{s\in S}\{s\}\times J_s,$$

where S is some topological space and for each $s\in S$ $J_s=(m_s, n_s)$ is an open interval containing 0.

(2) If $t_1\in J_s$ and $t_1+t_2\in J_s$, then $\pi((s, t_1), t_2)=(s, t_1+t_2)$.

(One may notice that by the assumption that π is a local system on X, the domain $\mathcal{D}$ of π is open in $X\times R$ and hence X is open in $S\times R$).

If in particular for each $s\in S, J_s=R$, i.e., $X=S\times R$, we say that π is a (*global*) *parallel flow*.

Let S be a topological space and $f: S\to R^+$ a positive continuous function. We define an equivalence relation $\sim$ on $S\times R$ by

$$(s_1, t_1)\sim(s_2, t_2) \Leftrightarrow s_1=s_2 \text{ and } t_1\equiv t_2 \bmod (f(s_1)).$$

The quotient space $(S\times R)/\sim$ is denoted by Y. We can assume, without loss of generality,

$$Y=\bigcup_{s\in S}\{s\}\times[0, f(s)),$$

but endowed with the quotient topology of $(S\times R)/\sim$.

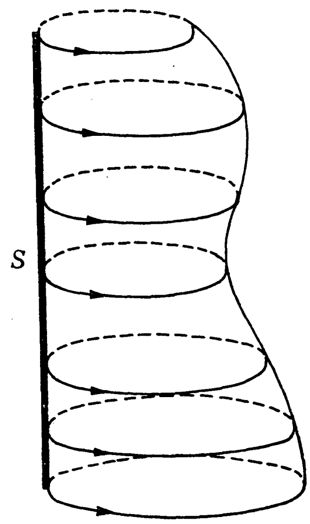

Fig. 1

Define $\rho : Y \times R \rightarrow Y$ by

$$\rho((s, r), t) = (s, t') \quad \text{for } (s, r) \in Y \text{ and } t \in R,$$

where $0 \leqq t' < f(s)$ and $t' \equiv t + r \mod (f(s))$. Then ρ is obviously a dynamical system on Y.

Definition 1.3. The dynamical system ρ on Y constructed in the manner described is called a *cylindroidally parallel flow* (Fig. 1).

§ 2. General Remarks on Invariant Positive Measures.

Definition 2.1. Let π and ρ be local systems on X and on Y, respectively, and let h be a homeomorphism of X onto Y. Then h is called an *isomorphism of* π *onto* ρ (denoted by $h : \pi \rightarrow \rho$) if the following condition (*) is satisfied:
(*) The equality

$$h(\pi(x, t)) = \rho(h(x), t)$$

holds whenever the left side is defined. (See [8]. In terms of [8], (h, Identity : $R \rightarrow R$) is a type 0 GH-isomorphism.)

We consider measures μ on X such that μ is a Borel measure and $\mu(\{x\}) = 0$ for every $x \in X$ (so that if X is second countable, then μ is a Lebesgue measure) and that μ satisfies the following conditions (1) and (2):

(1) *If* $G \subset X$ *is a non-empty open subset, then* $\mu(G) > 0$.

(2) *If* $K \subset X$ *is a compact subset, then* $\mu(K) < \infty$.

In the sequel, a "*measure*" or a "*positive measure*" is always assumed to satisfy (1) and (2).

Definition 2.2. Let π be a local system on X. A positive measure on X is called an *invariant positive measure with respect to* π (or we say that π *admits an invariant positive measure* μ) if for every measurable set $A \subset X$ and for every $t \in R$, we have

$$\mu(\pi(A, t)) = \mu(A),$$

whenever $\pi(x, t)$ is defined for all $x \in A$.

If further the total measure is finite, then μ is said to be *finite*.

Proposition 2.3. *Let* π *and* ρ *be local systems on* X *and on* Y, *respectively,* $h : \pi \rightarrow \rho$ *an isomorphism, and* μ_Y *an invariant positive measure with respect to* ρ. *For any subset* A *of* X *such that* $h(A)$ *is measurable with respect to* ρ_Y, *we define*

$$\mu_X(A) = \mu_Y(h(A)).$$

Then μ_X *is an invariant positive measure with respect to* π *and is called the induced invariant measure on* X *by* μ_Y *and* h.

Proof. Since a homeomorphism preserves openness, compactness and taking arbitrary intersections or arbitrary unions (and hence preserves Borel sets), μ_X is a positive measure. Further, since $h: \pi \rightarrow \rho$ is an isomorphism, for every Borel set $A \subset X$ and for every $t \in \bigcap_{x \in A} I_x$, $\rho(h(x), t)$ is defined for all $x \in A$ and we have

$$\begin{aligned}\mu_X(\pi(A, t)) &= \mu_Y(h(\pi(A, t))) \\ &= \mu_Y(\rho(h(A), t)) \\ &= \mu_Y(h(A)) = \mu_X(A).\end{aligned}$$

Hence μ_X is an invariant positive measure with respect to π.

Proposition 2.4. *Let π be a local parallel flow on $X = \bigcup_{s \in S} \{s\} \times J_s$ or a cylindroidally parallel flow on $X = \bigcup_{s \in S} \{s\} \times [0, f(s))$, where S admits a finite positive measure ν, and let $F: S \rightarrow [1, \infty)$ be a measurable function. We define a new measure μ on X by*

$$\mu(A) = \int_S \int_{-\infty}^{\infty} \frac{\phi_A(s, r)}{F(s)} dr d\nu$$

for each measurable set $A \subset X$ (with respect to a product measure on $(S \times R) \cap X$ or $(S \times R^+) \cap X$). Then μ is an invariant positive measure with respect to π.

Proof. We assume that π is a parallel flow. Since μ is obviously a positive measure, we have only to prove that μ is invariant with respect to π. In order to do this, it is enough to show that for each $A \subset X$ of the form

$$A = S_1 \times [t_1, t_2) \subset X$$

(where $S_1 \subset S$ is a measurable set and $t_1 \leqq t_2$), we have

$$\mu(A) = \mu(\pi(A, t)),$$

if $\pi(x, t)$ is defined for all $x \in A$. Put

$$\psi_A(s) \equiv \int_{-\infty}^{\infty} \phi_A(s, r) dr.$$

Then we have

$$\psi_A(s) = \begin{cases} t_2 - t_1 & s \in S_1 \\ 0 & s \notin S_1 \end{cases}$$

and

$$\mu(A) = \int_S \frac{\psi_A(s)}{F(s)} d\nu.$$

Since $\pi(A, t) = S_1 \times [t_1 + t, t_2 + t)$, we have

$$\psi_{\pi(A,t)}(s)=\begin{cases} t_2-t_1 & s\in S_1 \\ 0 & s\notin S_1. \end{cases}$$

Accordingly,

$$\mu(\pi(A,t))=\int_S \frac{\psi_{\pi(A,t)}(s)}{F(s)}d\nu$$
$$=\int_S \frac{\psi_A(s)}{F(s)}d\nu=\mu(A).$$

Hence μ is invariant with respect to π. The same proof is valid for cylindroidally parallel flows.

Corollary 2.5. *If $J_s=(m_s, n_s)$ is a finite interval (or $f(s)$ is finite) for all $s\in S$, then π admits a finite invariant positive measure such that $\mu(X)\leqq\nu(S)$.*

Proof. Put $F(s)=n_s-m_s+1$ or $F(s)=f(s)+1$. Then $F(s)$ is lower semicontinuous ([1], [9]), *a fortiori*, measurable and $1<F(s)<\infty$. Define μ as in Proposition 2.4. Then μ is an invariant measure. Further, since $\psi_X(s)=n_s-m_s$ or $\psi_X(s)=f(s)$, we have

$$\mu(X)=\int_S\int_{-\infty}^{\infty}\frac{\phi_X(s,r)}{F(s)}dr\,d\nu=\int_S\frac{\psi_X(s)}{F(s)}d\nu$$
$$\leqq\int_S d\nu=\nu(S).$$

Hence μ is a finite invariant positive measure with respect to π.

Proposition 2.6. *Let π be a local system on X, and $\{U_i\}$ be a countable open covering of X such that each U_i is quasi-invariant. If $\pi\|U_i$ admits an invariant positive measure μ_i for each i such that for every compact set $K\subset X$, there exists M_K, $0<M_K<\infty$, satisfying*

$$\mu_i(K\cap U_i)\leqq M_K \qquad \textit{for all } i,$$

then π admits an invariant positive measure μ such that

$$\mu(K\cap X)\leqq M_K.$$

If further $\mu_i(U_i)\leqq 1$ for all i, then we have $\mu(X)\leqq 1$.

Proof. Define μ by

$$\mu(A)=\sum_{n=1}^{\infty}\frac{1}{2^n}\mu_n(A\cap U_n) \quad \text{for every Borel set} \quad A\subset X,$$

then μ is obviously a positive measure and

$$\mu(K\cap X)\leqq\sum_{n=1}^{\infty}\frac{1}{2^n}\mu_n(U_n\cap K)\leqq M_K\sum_{n=1}^{\infty}\frac{1}{2^n}=M_K.$$

If further $\pi(x,t)$ is defined for all $x\in A$, then

$$\mu(\pi(A,t))=\sum_{n=1}^{\infty}\frac{1}{2^n}\mu_n(\pi(A\cap U_n,t))$$
$$=\sum_{n=1}^{\infty}\frac{1}{2^n}\mu_n(A\cap U_n)=\mu(A).$$

Hence μ is an invariant positive measure with respect to π. The second assertion is obvious.

§3. Existence of Invariant Positive Measures for Some Special Flows.

In the sequel we assume that X is R^2 or S^2 or an open subset of R^2. Since continuous sections in X are arcs ([6]), we easily obtain Propositions 3.1 and 3.4 to follow (cf. [1] and [2], respectively).

Proposition 3.1. *Let π be a completely unstable flow on X. Then for each $x\in X$ there exist a quasi-invariant open neighborhood U of x, a local parallel flow ρ on $Y=\bigcup_{s\in S}\{s\}\times J_s$ and an isomorphism $h:\pi\|U\to\rho$, where $S\subset U$ is homeomorphic to $(0,1)\subset R$ and J_s is the domain of $\pi\|U(s,\cdot)$.*

Remark 3.2. The set S in Proposition 3.1 is a section of $\pi\|U$ and the mapping $h:U\to Y$ is represented as follows:

$$h(x)=(s,t)\in Y \qquad \text{for } x\in U,$$

where $\pi(s,t)=x$. Accordingly for every $V\subset U$, we have

$$h(V)=\{(s,t);\ \pi(s,t)\in V\}.$$

If $h(x_0)=(s_0,t_0)$ and there exists $\alpha>0$ such that for all $|t|>\alpha, \pi(x_0,t)\notin V$, then we have

$$h(V)_{s_0}\equiv\{t\ ;\ \pi(s_0,t)\in V\}\subset[t_0-\alpha,t_0+\alpha]\subset R.$$

In the sequel, S denotes the open interval $(0,1)$ $(\subset R)$, endowed with the ordinary Lebesgue measure.

Corollary 3.3. *If π is a completely unstable flow on X, and if $I_x=(a_x,b_x)$ is a finite interval for all $x\in X$, then π admits a finite invariant positive measure.*

Proof. By Proposition 3.1, there exist an open covering $\{U_i\}$ of X and parallel flow ρ_i on Y_i and isomorphisms h_i such that for each $i, h_i:\pi\|U_i\to\rho_i$, where $Y_i=\bigcup_{s\in S}\{s\}\times J_s^i, J_s^i\subset R$ is a finite open interval containing 0. By Corollary 2.5, ρ_i admits an invariant positive measure ν_i such that $\nu_i(Y_i)\leqq\nu(S)=1$. Let μ_i be the induced measure on U_i by ν_i and h_i. Then $\mu_i(U_i)\leqq1$. Hence by Proposition 2.6, π admits a finite invariant positive measure.

Proposition 3.4. *Let π be a local system on X. If all points in X are non-singular periodic, then for each $x\in X$, there exists an invariant open neigh-*

borhood U *of* x, *a cylindoroidally parallel flow* ρ *on* $Y=\bigcup_{s\in S}\{s\}\times[0, T(s))$ *and an isomorphism* $h: \pi\|U\to\rho$, *where* $S\subset U$.

Corollary 3.5. *Let* π *be a local system on* X. *If all points in* X *are non-singular periodic, then* π *admits a finite invariant positive measure.*

Proof. Combining Corollary 2.5 and Proposition 2.6 with Proposition 3.4, the proof is done in the same way as above.

§4. Local Existence of Invariant Positive Measures.

Before discussing existence of invariant measures on X, we treat the problem locally. First, we shall introduce the notion of a local invariant measure.

Definition 4.1. Let π be a local system on X, and $x\in X$. We say that π *admits an invariant positive measure locally at* x if there exists an open neighborhood U of x such that $\pi\|U$ admits a finite invariant positive measure.

Theorem 4.2. *Let* π *be a local system on* X. *If* $x\in X$ *is regular, then* π *admits an invariant positive measure locally at* x.

Proof. It is known ([9]) that there exists an open neighborhood U of x such that $\pi\|U$ is isomorphic to a local parallel flow ρ on $Y=S\times(-\alpha,\alpha)$, where α is some positive number. By Proposition 2.3 and Corollary 2.5, $\pi\|U$ admits a finite invariant positive measure.

As recalled in the Introduction, the following is known ([7]).

Proposition 4.3. *Let* π *be a local system, and* $x\in X$ *an isolated singular point. If* π *admits an invariant positive measure locally at* x, *then* x *is a center or a generalized saddle.*

The main purpose of this section is to establish the converse, i.e.,

Theorem 4.4. *Let* π *be a local system on* X. *If* x *is a center or a generalized saddle, then* π *admits an invariant positive measure locally at* x.

Proof of the Theorem for a center x. If x is a center, then there exists an open neighborhood U of x such that all points in $U-\{x\}$ are non-singular periodic. By Corollary 3.5, $\pi\|(U-\{x\})$ admits a finite invariant positive measure μ'. Put

$$\mu(A)=\mu'(A-\{x\}) \quad \text{for every Borel set } A\subset U.$$

Then μ has the desired properties.

Next we treat generalized saddles. we need the following lemma, which is a direct consequence of the definition of a generalized saddle.

Lemma 4.5. *Let* π *be a local system on* X *and* x *a generalized saddle. Then there exists an open neighborhood* U *such that* $\pi\|(U-C_x)$ *satisfies the conditions of Corollary* 3.3, *where* $C_x=\{y\,;\,L^+(y)=\{x\}$ *or* $L^-(y)=\{x\}\}$.

Proof of the Theorem for a generalized saddle x. Let U be an open neighborhood of x as in Lemma 4.5. By Corollary 3.3, $\pi\|(U-C_x)$ admits a finite invariant positive measure μ'. Define μ by

$$\mu(A)=\mu'(A-C_x) \qquad \text{for every Borel set } A\subset U.$$

then μ has the desired properties.

Remark 4.6. Measures constructed above are Lebesgue measures, but not ordinary one on R^2.

§ 5. Global Existence of Invariant Positive Measures.

We restrict our consideration to a local system π on R^2 for which there are only a finite number of singular points. In this section, we shall seek necessary and sufficient conditions for existence of invariant positive measures with respect to π.

Lemma 5.1. *If there exists a relatively compact open subset* $U\subset R^2$ *such that* $C^+(U)\subset\bar{U}$ *and* $C^-(U)\not\subset\bar{U}$ *(or* $C^-(U)\subset\bar{U}$ *and* $C^+(U)\not\subset\bar{U}$*), then* π *admits no invariant positive measure.*

Proof. We assume the contrary, i.e., π admits an invariant positive measure μ. By symmetry we can assume that $C^+(U)\subset\bar{U}$ and $C^-(U)\not\subset\bar{U}$. Then, by the continuity of π, there exist an open subset $V\subset U$ and $t>0$ such that $\pi(V,-t)\cap U=\phi$. Since μ is invariant and $\pi(\pi(V,-t)\cup U,t)\subset U$, we have

$$\begin{aligned}\infty>\mu(U)&\geqq\mu((\pi(V,-t)\cup U,t))=\mu(\pi(V,-t)\cup U)\\&=\mu(V,-t)+\mu(U)=\mu(V)+\mu(U).\end{aligned}$$

This is a contradiction to $\mu(V)>0$.

Theorem 5.2. *Let* π *be a local system on* R^2 *and assume that there are only a finite number of singular points. Then the following statements* (1), (2) *and* (3) *are equivalent*:

(1) *π admits an invariant positive measure.*

(2) (i) *Every singular point is a center or a generalized saddle.*

(ii) *For a non-periodic point* $x\in X$, $L^+(x)$ *and* $L^-(x)$ *each contains at most one point.*

(3) $C=\{x\in X;$ *non-periodic and either* $L^+(x)\neq\phi$ *or* $L^-(x)\neq\phi\}$ *consists of a finite number of orbits.*

Proof. (2) $\Leftrightarrow$ (3) is immediate. We shall prove that (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2) : By the results in § 4, (2) (i) is obvious. In order to prove (2) (ii) by reductio ad absurdum, we assume that there exists a non-periodic point $x\in X$ such that $L^+(x)$ contains at least two points. Then $L^+(x)$ contains at least one regular point, say y, since we have only a finite number of singular points by our standing hypothesis ([6]). Let S be a local section at y. Since $y\in L^+(x)$, $C^+(x)$ intersects S at infinitely many points. Let s_1 and s_2 be arbitrary successive intersections of S and $C^+(x)$, and let U be the region surrounded by

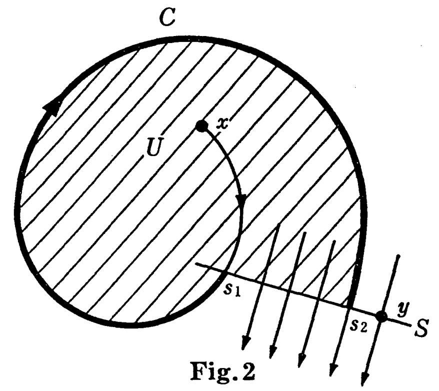

Fig. 2

a closed curve $\widehat{s_1Cs_2s_1}$ (Fig. 2). Then U obviously satisfies the conditions of Lemma 5.1. This is a contradiction to our assumption. For the negative limit set, we can prove similarly.

(2) $\Rightarrow$ (1) : Let $\mathcal{S}$ and $\mathcal{P}$ denote the set of singular points and the set of periodic points, respectively. Put

$$M=R^2-\mathcal{P} \quad \text{and} \quad P_0=\mathcal{P}-\mathcal{S}.$$

Let N and M_0 be defined by

$$N=\{x\in M\,;\, L^+(x)\neq\phi \quad \text{or} \quad L^-(x)\neq\phi\}$$

and

$$M_0=R^2-\bar{P}_0-\mathcal{S}-N.$$

Since the fact that $L^+(x)$ $(L^-(x))$ is a singleton implies that $L^+(x)$ $(L^-(x))\subset\mathcal{S}$, we can easily see that N consists of a finite number of orbits. (Recall that every singular point is a center or a generalized saddle).

In order to prove (2) $\Rightarrow$ (1), we need several lemmas. To simplify the discussion, we shall assume here that π is a global system on R^2, but the same proof with slight modifications is valid for a local system π on R^2.

Lemma 5.3. *P_0 and M_0 are open in R^2 and $P_0\cup M_0$ is dense in R^2.*

Proof. Since $\mathcal{S}$ is closed in R^2 and since $\bar{N}\subset\mathcal{S}\cup N$ by assumption, M_0 is obviously open. We shall show that P_0 is open. Assume the contrary. Then there exists $C(x)\subset P_0$ such that for every neighborhood V of $C(x)$, we have

$$V\cap M\neq\phi.$$

Therefore we can easily see that either $C(x)$ itself is a limit cycle or there exists a non-singular limit cycle (cf. [6]), i.e., there exists $x\in M$ such that $L^+(x)$ or $L^-(x)$ is not a singleton. This is a contradiction to the assumption that for every $x\in M$, each of $L^+(x)$ and $L^-(x)$ contains at most one point. Hence P_0 is open. The second assertion is obvious, since $R^2=P_0\cup M_0\cup\mathcal{S}\cup N\cup(\bar{P}_0-P_0)$, $\mathcal{S}$ is a finite set and N consists of a finite number of orbits.

Lemma 5.4. *$\pi\|M$ is completely unstable and so is $\pi\|M_0$.*

Proof. To prove the lemma by reductio ad absurdum, assume that there exists $x \in M$ such that $x \in J^+(x)$. Then we can prove, as in the proof of (1) $\Rightarrow$ (2), that there exists an open set satisfying the conditions of Lemma 5.1. So there exist a infinitie number of non-periodic orbits such that the limit set (+ or −) is non-empty. Since N contains at most a finite number of orbits, this is a contradiction. Hence $x \notin J^+(x)$ for all $x \in M$, i.e., $\pi\|M$ is completely unstable. The second assertion is obvious.

By Lemma 5.4 and Proposition 3.1, there exist a countable open covering $\{V_i\}$ of M_0 such that every V_i is invariant with respect to π, a parallel flow ρ on $Y = S \times R$ and an isomorphism h_i for every i such that $h_i : \pi\|V_i \to \rho$. Let $V \subset R^2$ be a relatively compact open set. Put

$$F_i^V(s) = \int_{-\infty}^{\infty} \phi_{h_i(V \cap V_i)}(s, r)dr \quad \text{for each } i,$$

where $\phi_{h_i(V \cap V_i)}$ is the characteristic function of $h_i(V \cap V_i) \subset Y$.

Lemma 5.5. *For each i, F_i^V is a measurable function of S into $[0, \infty)$. If further there exists $T>0$ such that for all $|t|>T$,*

$$\pi(V, t) \cap V = \phi,$$

then $F_i^V(s) < 2T$ for all $s \in S$ and i.

Proof. Since $L^+(x) = \phi$ for all $x \in M_0$, there exists T_x such that $\pi(x, t) \notin V$ for all $|t| > T_x$. By Remark 3.2,

$$F_i^V(s) = \int_{-\infty}^{\infty} \phi_{h_i(V \cap V_i)}(s, r)dr \leqq 2T_x,$$

where $h_i(x) = (s, t_0)$. Hence F_i^V is a mapping of S into $[0, \infty)$. Further, since $h_i(V \cap V_i)$ is open in Y, F_i^V is measurable by Fubini's Theorem. The second assertion is obvious.

Lemma 5.6. *$\pi\|M_0$ admits an invariant positive measure μ such that for every compact set $K \subset R^2$*

$$(*) \qquad \mu(K \cap M_0) < \infty.$$

Proof. Let V_0 be a relatively compact open neighborhood of $\mathcal{S}$ in R^2. For each i, define an invariant positive measure ν_i on Y by

$$\nu_i(A) = \int_0^1 \int_{-\infty}^{\infty} \frac{\phi_A(s, r)}{F_i^{V_0}(s) + 1} drds$$

for every Borel set $A \subset Y$. By Lemma 5.5 and Proposition 2.4, ν_i is invariant with respect to ρ. Let μ_i be the invariant measure on V_i induced by ν_i and

h_i. We shall show that μ_i satisfies the condition in Proposition 2.6, i.e., for every compact set $K \subset R^2$, there exists a constant α_K such that

$$\mu_i(K \cap M_0) < \alpha_K \quad \text{for all } i.$$

If $x \in M \cap K$, since $x \notin J^+(x)$ by Lemma 5.4, there exist an open neighborhood $V(x)$ and a positive number T_x such that for all $|t| > T_x$ we have

$$\pi(V(x) \cap M_0, t) \cap V(x) = \phi.$$

Then $\{V(x), V_0, P_0\}_{x \in M \cap K}$ is an open covering of K in R^2. Since K is compact, we can assume

$$K \cap M_0 \subset \bigcup_{j=1}^{n} V(x_j) \cup V_0.$$

Put

$$T = \operatorname*{Max}_j \{T_{x_j}\}.$$

Then for each i, we have

$$\begin{aligned}\mu_i(V_0 \cap V_i) &= \nu_i(h_i(V_0 \cap V_i)) \\ &= \int_0^1 \int_{-\infty}^{\infty} \frac{\phi_{h_i(V_0 \cap V_i)}(s, r)}{F_i^{V_0}(s)+1} drds \\ &= \int_0^1 \frac{F_i^{V_0}(s)}{F_i^{V_0}(s)+1} ds \leqq 1\end{aligned}$$

and

$$\begin{aligned}\mu_i(V(x_j) \cap V_i) &= \int_0^1 \int_{-\infty}^{\infty} \frac{\phi_{h_i(V(x_j) \cap V_i)}(s, r)}{F_i^{V_0}(s)+1} drds \\ &= \int_0^1 \frac{F_i^{V(x_j)}(s)}{F_i^{V_0}(s)+1} ds \qquad (j=1, 2, \cdots, n).\end{aligned}$$

By Lemma 5.5, we have $F_i^{V(x_j)}(s) \leqq 2T$ $(j=1, 2, \cdots, n)$ for all $s \in S$. Hence we have

$$\mu_i(V(x_j) \cap V_i) \leqq 2T \qquad (j=1, 2, \cdots, n).$$

Consequently,

$$\begin{aligned}\mu_i(K \cap V_i) &\leqq \mu_i(V_0 \cap V_i) + \sum_{j=1}^{n} \mu_i(V(x_j) \cap V_i) \\ &\leqq 1 + 2nT \qquad (i=1, 2, \cdots, n).\end{aligned}$$

Therefore, by Proposition 2.6, $\pi \| M_0$ admits an invariant positive measure μ satisfying $(*)$.

Proof of (1) $\Rightarrow$ (2) in Theorem 5.2. By Corollary 3.5 and Lemma 5.6,

$\pi||P_0$ and $\pi||M_0$ admit invariant positive measures μ^1 and μ^2, respectively, where μ^1 is finite and μ^2 satisfies $(*)$ in Lemma 5.6.

Define μ by

$$\mu(A)=\mu^1(A\cap P_0)+\mu^2(A\cap M_0)$$

for each Borel set $A\subset R^2$. Then μ is obviously invariant. Further, by Lemma 5.3, for a non-empty open set $G\subset R^2$, we have $G\cap P_0\neq\phi$ or $G\cap M_0\neq\phi$. Hence $\mu(G)>0$. Since μ^1 is finite and μ^2 satisfies $(*)$ in Lemma 5.6, we have

$$\mu(K)=\mu^1(K\cap P_0)+\mu^2(K\cap M_0)<\infty,$$

for every compact set $K\subset R^2$. Consequently μ is an invariant positive measure.

Lemma 5.7. *Let π be a global parallel flow on $X=S\times R$. Then π does not admit any finite invariant positive measure.*

Proof. Assume that π admits a finite positive invariant measure μ and put $S_n=S\times[n-1, n)$ $(n=1, 2, 3, \cdots)$. Then we have $\pi(S_1, n-1)=S_n$ and $S_n\cap S_m=\phi$ for $n\neq m$. Hence we have

$$\infty>\mu(X)\geqq\mu\left(\bigcup_{n=1}^{\infty}S_n\right)=\sum_{n=1}^{\infty}\mu(S_n)=\infty.$$

This is a contradiction.

Using Lemma 5.7 just proved, we have the following

Corollary 5.8. *Let π be a global dynamical system on X. If there exists an open subset $U\subset X$ such that $\pi||U$ is parallelizable, then π does not admit any finite invariant positive measure.*

Theorem 5.9. *Let π be a global dynamical system on R^2 and assume that there are only a finite number of singular points. Then π admits a finite invariant positive measure if and only if the number of non-periodic orbits is finite.*

Proof. We shall use notations in Theorem 5.2. Sufficiency: M_0 contains only a finite number of orbits by the assumption and M_0 is open in R^2 by Lemma 5.3. Hence we have $M_0=\phi$, i.e., P_0 is dense in R^2. By Corallary 3.5, $\pi||P_0$ admits a finite invariant positive measure. Consequently, π admits a finite invariant positive measure.

Necessity: Assume the contrary, i.e., π has an infinite number of non-periodic orbits. By Corollary 5.8, we have $M_0=\phi$, i.e., $R^2=\bar{P}_0\cup\mathcal{S}\cup N$. Since $\mathcal{S}\cup N$ consists of a finite number of orbits by Theorem 5.2, P_0 is dense in R^2 and $H=\bar{P}_0-P_0-N-\mathcal{S}$ contains an infinite number of orbits. If $x\in H$, $C(x)$ separates R^2 into two regions, each of which contains at least one periodic orbit, because $L^{\pm}(x)=\phi$ and P_0 is dense in R^2. Hence by induction, we can easily see that R^2-H has an infinite number of connected components, each of which contains at least one periodic orbit. Since the bounded region surrounded by a periodic orbit contains at least one singular point, we can conclude that π has an infinite number of singular points. This is a contradiction to our standing assumption.

§6. An Invariant Positive Measure on S^2.

Let π be a dynamical system on S^2, so that π is global, and assume that there are only a finite number of singular points. Since S^2 is compact, we shall consider only finite positive measures. Let $x_0 \in S^2$ be a singular point. Then there exists a homeomorphism $h : S^2 - \{x_0\} \to R^2$. Define a dynamical system π^* on R^2 by

$$\pi^*(x, t) = h(\pi(h^{-1}(x), t))$$

for all $x \in R^2$, then h is an isomorphism $\pi || \{S^2 - \{x_0\}\} \to \pi^*$.

Proposition 6.1. *π admits an invariant positive measure if and only if π^* admits a finite invariant positive measure.*

Proof. Easy.

Combining this Proposition with Theorem 5.9, we obtain the following:

Theorem 6.2. *Let π be a dynamical system on S^2, and assume that there are only a finite number of singular points. Then π admits a finite invariant positive measure if and only if the number of non-periodic orbits is finite.*

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