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Cauchy Problem for Hyperbolic Systems in L^p

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§1. Introduction.

This paper concerns the Cauchy problem for hyperbolic systems in L^p, $p\neq 2$, $1\leqq p\leqq\infty$,

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}=\sum_{j=1}^{n} A_j\dfrac{\partial u}{\partial x_j}+B\cdot u \\ u(0,x)=u_0(x), \quad (t,x)\in[0,T]\times R^n. \end{cases}$$

The corresponding problem for wave equation has been treated by Littman [4]. Also, Brenner [1] has investigated the same problem for symmetric hyperbolic systems, whose results contained one of Littman. Brenner has proved that the problem (1) for symmetric hyperbolic systems is well posed in $L^p(p\neq 2)$, if and only if the matrices A_j commute. We prove that the same result as Brenner's holds under weaker condition than strong hyperbolicity for (1). We say that (1) is a strongly hyperbolic system if for all $\xi=(\xi_1,\cdots,\xi_n)\in R^n-\{0\}$, the eigenvalues of $A(\xi)=\sum_{j=1}^{n} A_j\xi_j$ are real and $A(\xi)$ is uniformly diagonalizable in ξ. It is well known that (1) is strongly hyperbolic if and only if it is well posed in L^2 [7]. From our result, we think that function space L^2, together with function space $\mathcal{E}$ [5], is suitable for hyperbolic equations.

Our main theorem (§3) follows by application of techniques developed by Hörmander [2] and Brenner [1], and a matrix lemma (§2).

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§2. A matrix lemma.

Matrices, for which any complex linear combination $\alpha A+\beta B$ has as characteristic roots the numbers $\alpha\lambda_i+\beta\mu_i$, are said to have property L [6]. Here, we prove a lemma for matrices with property L which is useful for proof of the theorem.

Lemma. *Let $M_j(j=1,\cdots,l)$ be $N\times N$ matrices such that the next two conditions hold,*

(i) *for all $\alpha=(\alpha_1,\cdots,\alpha_l)\in R^l-\{0\}$, the eigenvalues of $M(\alpha)=\sum_{j=1}^{l} M_j\alpha_j$ are real and $M(\alpha)$ is diagonalizable,*

(ii) *the eigenvalues of* $M(y)=\sum_{j=1}^{l} M_j y_j$ *for all* $y=(y_1,\cdots,y_l)$ *in an open non void ball* K *in* R^l *are of the form*

$$\sum_{j=1}^{l} \alpha_{kj} y_j \qquad (k=1,\cdots,N)$$

where the α_{kj} *are constant.*

Then, the matrices $M_1,\cdots,M_l$ *commute.*

Proof. Without loss of generarity, we have only to prove the lemma when $l=2$. From the regularity of the both members in the equality

$$\det\left(\lambda I-\sum_{j=1}^{2} M_j y_j\right)=\prod_{k=1}^{N}\left(\lambda-\sum_{j=1}^{2}\alpha_{kj}y_j\right),\quad y\in K,$$

we see that it is also satisfied for all $y=(y_1, y_2)\in \boldsymbol{C}^2$.

We set $M(\kappa)=M_1+\kappa M_2$. By the assumptions, the eigenvalues of $M(\kappa)$ are of the form $\lambda_j(\kappa)=\alpha_{j1}+\kappa\alpha_{j2}$ $(j=1,\cdots,N)$. Let $\lambda_j(\kappa)$ be an eigenvalue with constant multiplicity, and $P_{m_j}(\kappa)$ be an eigenprojection associated with $\lambda_j(\kappa)$. To prove $M_1M_2=M_2M_1$, we investigate the regularity of eigenprojections $P_{m_j}(\kappa)$ for $\lambda_j(\kappa)$.

① $\mathrm{Im}\,\kappa\neq 0$.

Multiplicity of the eigenvalues of $M(\kappa)$ is constant. For, we assume that $\lambda_i(\kappa_0)=\lambda_j(\kappa_0)$ for some $\kappa_0\in\boldsymbol{C}$, $\mathrm{Im}\,\kappa_0\neq 0$. Let κ_0 be $\kappa_{01}+i\kappa_{02}$ (κ_{01},κ_{02} are real and $\kappa_{02}\neq 0$).

$$\begin{aligned}\lambda_i(\kappa_0)&=\alpha_{i1}+\kappa_0\alpha_{i2}=(\alpha_{i1}+\kappa_{01}\alpha_{i2})+i\kappa_{02}\alpha_{i2}\\&=\lambda_j(\kappa_0)=\alpha_{j1}+\kappa_0\alpha_{j2}=(\alpha_{j1}+\kappa_{01}\alpha_{j2})+i\kappa_{02}\alpha_{j2}\\&\therefore\quad \alpha_{i1}=\alpha_{j1},\quad \alpha_{i2}=\alpha_{j2}\\&\therefore\quad \lambda_i(\kappa)=\lambda_j(\kappa);\quad \kappa\in\boldsymbol{C}\text{ and }\mathrm{Im}\,\kappa\neq 0.\end{aligned}$$

Therefore, multiplicity is constant. Then, from Kato [3; pp. 63~74], eigenprojection $P_{m_j}(\kappa)$ is holomorphic function of κ in $\boldsymbol{C}^+=\{z \mid \mathrm{Im}z>0\}$ and $\boldsymbol{C}^-=\{z \mid \mathrm{Im}\,z<0\}$.

② $\mathrm{Im}\,\kappa=0$.

By the assumptions, M_1 is diagonalizable and $\lambda_j(\kappa)$ has the form $\alpha_{j1}+\kappa\alpha_{j2}$ $(j=1,\cdots,N)$. So, by the theory of reduction process in Kato [3; pp. 81~83, pp. 85], $P_{m_j}(\kappa)$ is holomorphic at $\kappa=0$. The same is true at real κ since $M(\kappa)$ and $\lambda_j(\kappa)$ $(j=1,\cdots,N)$ are linear in κ, and $M(\kappa)$ is diagonalizable for any real κ. Thus, $P_{m_j}(\kappa)$ is holomorphic at any real κ.

③ $\kappa=\infty$.

$P_{m_j}(\kappa)$ is holomorphic at $\kappa=\infty$. For, M_2 is diagonalizable, and the eigenprojections of $M(\kappa)=\kappa(M_2+\kappa^{-1}M_1)$ coincide with of $M_2+\kappa^{-1}M_1$, to which the theory of reduction process applies to $\kappa=\infty$. So, $P_{m_j}(\kappa)$ is holomorphic at $\kappa=\infty$.

Hence, from ①, ② and ③, $P_{m_j}(\kappa)$ is holomorphic everywhere including $\kappa=\infty$, and so must be a constant by Liouville's theorem. Since M_1 and M_2 have common eigenprojections and both are diagonalizable, they commute. Thus lemma is proved.

Remark 1. This matrix lemma contains the results obtained by Motzkin and Taussky where M_1 and M_2 are Hermitian [6].

§ 3. Cauchy problem.

We first introduce some notations. If $v=(v_1, \cdots, v_N)$ and $u=(u_1, \cdots, u_N)$ are complex vectors, $\langle u, v\rangle$ will denote their scalar product and $|v|$ the Euclidean norm,

$$\langle u, v\rangle=\sum_{j=1}^{N} u_j\bar{v}_j, \quad |v|=\left(\sum_{j=1}^{N}|v_j|^2\right)^{1/2}.$$

If $v_j\in\mathcal{S}(R^n)$ for $j=1, \cdots, N$, then we say that $v=(v_1, \cdots, v_N)\in\mathscr{S}$. By $\mathcal{L}^p$ we mean the set of functions $v=(v_1, \cdots, v_N)$ with $v_j\in L^p(R^n)$ $(j=1, \cdots, N)$ and for $p<+\infty$, we set

$$\|v\|_p=\left(\int_{R^n}|v(x)|^p dx\right)^{1/p}$$

and for $p=\infty$

$$\|v\|_\infty=\text{ess sup}\{|v(x)| : x\in R^n\}.$$

We now turn to the Cauchy problem. Let $A_j(j=1, \cdots, n)$ and B be $N\times N$ constant matrices and let $u(t, x)$ and $u_0=u_0(x)$ be N-dimensional complex vector functions. We consider the Cauchy problem

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t}=\sum_{j=1}^{n} A_j\dfrac{\partial u}{\partial x_j}+B\cdot u \\ u(0, x)=u_0(x), \quad (t, x)\in[0, T]\times R^n. \end{cases}$$

We assume that the next condition (I) holds for (1),

Condition (I) : for all $\xi=(\xi_1, \cdots, \xi_n)\in R^n-\{0\}$, the eigenvalues of $A(\xi)=\sum_{j=1}^{n} A_j\xi_j$ are real and $A(\xi)$ is diagonalizable.

Definition. The Cauchy problem (1) is well posed in L^p if for each $u_0\in\mathscr{S}$, there is a solution $u=u(t, x)$ of (1) in $\mathcal{L}^p$ norm which satisfies the inequality

$$\|u(t, \cdot)\|_p\leqq C(T)\|u_0\|_p, \quad 0\leqq t\leqq T.$$

Such a solution is unique.

Remark 2. For $p=\infty$, the above definition of well-posedness is weaker than the usual one since $\mathcal{S}$ is not dense in $\mathcal{L}^\infty$.

Theorem. *Suppose* $p\neq 2$, $1\leqq p\leqq\infty$. *Then under the condition* (I), *the Cauchy problem* (1) *is well posed in* L^p *if and only if the matrices* $A_1,\cdots,A_n$ *commute.*

The proof of the theorem is similar to Brenner's [1] and is ommited.

By the remark 2, this theorem gives a necessary condition for the usual definition of well-posedness in L^∞.

Remark 3. Under the condition (I), the matrices $A_1,\cdots,A_n$ commute if and only if $A_1,\cdots,A_n$ are diagonalizable simultaneously. Consequently, if the Cauchy problem (1) is well posed in $L^p(p\neq 2)$ under the condition (I), (1) is strongly hyperbolic.

Remark 4. When we treat non-linear hyperbolic equations with more than one space variable, we think that the above theorem is useful for the choice of function spaces for the Cauchy problem of those equations.

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