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Connectedness Properties of Start Points in Semi-Dynamical Systems

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0. Introduction.

In [2], we discussed some properties of start points in products of semi-dynamical systems. Notions of proper/improper start points were introduced and a criterion established for a point to be an improper start point. Necessary and sufficient conditions for sets of proper/improper start points to be dense everywhere were obtained. In this paper, we discuss the connectedness properties of these sets. This paper is divided into two sections. In the first section we prove that, in a semi-dynamical system (X, π), set of start points does not disconnect an open connected set; and the (path) connectedness of X is equivalent to that of $X-S$. The second section is devoted to products of semi-dynamical systems, and we examine connectedness and path connectedness of the sets of proper/improper start points and of the set of start points. In the presence of just two start points in different factor semi-dynamical systems, (path) connectedness of product space is equivalent to (path) connectedness of the set of proper start points. The case when only one of the factor systems contains a start point is also considered. Further in the presence of an improper start point, (path) connectedness of the set of start points or the set of improper start points is equivalent to (path) connectedness of the product space. Results on path connectedness are in contrast with the situation in topology where closure of a path connected set is not necessarily path connected; indeed in a topological space X if $A \subset K \subset \mathrm{Cl} A$ and both A, $\mathrm{Cl} A$ are path connected, the set K is not necessarily path connected. Further [5, §31.6] similar theorems on arc connectedness hold for Hausdoroff spaces.

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Notation and definitions used are the same as in [2, §1].

1. Some Basic Theorems.

Lemma 1.1. *Let* (X, π) *be a semi-dynamical system. Let* U *be an open*

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subset of X and x_λ a net in U such that $x_\lambda \to x \in U$. Define a function $E: U \to R^+ \cup \{+\infty\}$ by $E(x) = \sup\{\theta \geqq 0 : x[0, \theta] \subset U\}$, $x \in U$. Then there exists $T > 0$ such that $\inf_{\lambda \geqq \lambda_0} E(x_\lambda) \geqq T$ for some λ_0.

Proof. Suppose not; then there exists a sub-net $E(x_\lambda)$ converging to zero. Without loss of generality, let $E(x_\lambda) \to 0$. Since U is open, $x_\lambda E(x_\lambda) \in \partial U$, so that $x_\lambda E(x_\lambda) \to x0 = x \in \partial U$. But $x \in U$ and U is open; contradiction.

Theorem 1.2. *In a semi-dynamical system (X, π) the set of start points does not disconnect an open connected set.*

Proof. Let S be the set of start points and U an open connected sub-set of X. If possible, let $U-S$ be not connected. Then there exist disjoint non-empty sets G_1, G_2 open in $U-S$ such that $U-S = G_1 \cup G_2$. Define the function E as in the lemma. Then $U-S = \bigcup_{x \in U} x(0, E(x)) = G_1 \cup G_2$. Clearly, for every $x \in U$, $x(0, E(x))$ is contained in one of G_1, G_2 and does not intersect the other. Thus the sets $X_1 = \{x \in U : x(0, E(x)) \subset G_1\}$ and $X_2 = \{x \in U : x(0, E(x)) \subset G_2\}$ are disjoint. Moreover, $X_1 \cup X_2 = U$ and $X_1 \neq \phi, X_2 \neq \phi$. Then X_1 is open in U. Indeed, otherwise, there exists a net x_λ in X_2 such that $x_\lambda \to x \in X_1$. Then, making preliminary adjustments if necessary, $\inf_\lambda E(x_\lambda) \geqq T$ for some $T > 0$. Pick T' such that $0 < T' < \min(T, E(x))$. Then $x_\lambda T' \in G_2$, $xT' \in G_1$. But $x_\lambda T' \to xT'$. It contradicts that G_1, G_2 are open in $U-S$. Similarly X_2 is open in U. Thus X_1, X_2 form a disconnection of U; contradiction.

Theorem 1.3. *Let (X, π) be a semi-dynamical system. Let S be the set of start points. Then $X-S$ is (path) connected if and only if X is (path) connected.*

Proof. If X is connected, then connectedness of $X-S$ follows from the previous theorem.

Now let X be path connected [5, p. 197]. Let $x, y \in X-S$; then $x = x_1 T$, $y = y_1 T$ for some $T > 0$, and x_1, y_1 in $X-S$. If $f: I \to X$ is a path joining x_1 to y_1, then $\pi^T \circ f: I \to X-S$ is a path from x to y in $X-S$.

For the converse notice that $\mathrm{Cl}(X-S) = X$ and that points on a positive trajectory are path connected.

2. Products of Semi-Dynamical Systems.

Let $(X_\alpha, \pi_\alpha), \alpha \in a$ be a family of semi-dynamical systems and (X, π) the product [2, § 2.2] semi-dynamical system. Throughout S_α denotes the set of start points in (X_α, π_α); $S, S^*, S-S^*$, denote, respectively, the sets of start points, proper start points [2, Def. 2.5] and improper start points in (X, π).

Theorem 2.1. *Let (X_α, π_α), $\alpha \in a$ and (X, π) be as above. Let S_μ, S_ν be non-empty for some distinct μ and ν in a. Then the set of proper start points* *is (path) connected if and only if X is (path) connected.*

Proof. Let X be (path) connected so that X_α is (path) connected for each α. Let $z=\{z_\alpha\}_{\alpha\in a}$, $z'=\{z'_\alpha\}_{\alpha\in a}$ be proper start points so that $z_\rho\in S_\rho$ and $z'_\beta\in S_\beta$ for some ρ, β in a.

If $\rho\neq\beta$, consider the sets

$$K_1=\{z_\rho\}\times\prod_{\alpha\neq\rho}X_\alpha,\quad K_2=\{z'_\beta\}\times\prod_{\alpha\neq\beta}X_\alpha.$$

Clearly K_1, K_2 are (path) connected sub-sets of S^*. Moreover, $K_1\cap K_2\neq\phi$ and $z\in K_1, z'\in K_2$. Thus $K_1\cup K_2$ is a (path) connected sub-set of S^* and contains the points z, z'.

If $\rho=\beta$, one of $\beta\neq\mu$, $\beta\neq\nu$ must hold, say the former. Let $x_\mu\in S_\mu$. Consider the (path) connected sets

$$K_1=\{z_\beta\}\times\prod_{\alpha\neq\beta}X_\alpha,\quad K_2=\{z'_\beta\}\times\prod_{\alpha\neq\beta}X_\alpha \text{ and } K_3=\{x_\mu\}\times\prod_{\alpha\neq\mu}X_\alpha.$$

Since $K_1\cap K_3\neq\phi$, $K_2\cap K_3\neq\phi$, therefore, $K_1\cup K_2\cup K_3$ is a (path) connected sub-set of S^*. Moreover $z, z'\in K_1\cup K_2\cup K_3$. Hence S^* is (path) connected.

Conversely let S^* be (path) connected. We need show that X_α is (path) connected for every α. Let $\beta\in a$ be arbitrary and $z_\beta, z'_\beta\in X_\beta, z_\beta\neq z'_\beta$. One of $\beta\neq\mu$, $\beta\neq\nu$ must hold, say the former. Let $s_\mu\in S_\mu$. Pick x and y in X, $x=\{x_\alpha\}$, $y=\{y_\alpha\}$ such that $x_\mu=s_\mu=y_\mu$, $x_\beta=z_\beta$, and $y_\beta=z'_\beta$. Then x, y are proper start points. Since S^* is (path) connected, $\operatorname{proj}_\beta(S^*)\subset X_\beta$ is a (path) connected set containing z_β, z'_β. Hence X_β is (path) connected, etc.,

Corollary 2.2. *If X is connected and S_α is non-empty for infinitely many α in a, then S is connected.*

Proof. Here S^* is dense [2, Th. 2.13] in X, and, so, $S^*\subset S\subset X=\mathrm{Cl}S^*$ etc.,

Remark 2.3. The following example shows that (path) connectedness of S^* does not imply that of S_α for every $\alpha\in a$.

Let $X_1=[0,\infty)\subset R$. Define $\pi_1: X_1\times R^+\to X_1$ by $\pi_1(x,t)=x+t, x\in X_1, t\in R^+$.

Let $X_2=\{(x_1,0): x_1\geqq 0\}\cup\{(x_1,x_2): -1\leqq x_2\leqq 1, x_1=-|x_2|\leqq 0\}\subset R^2$.

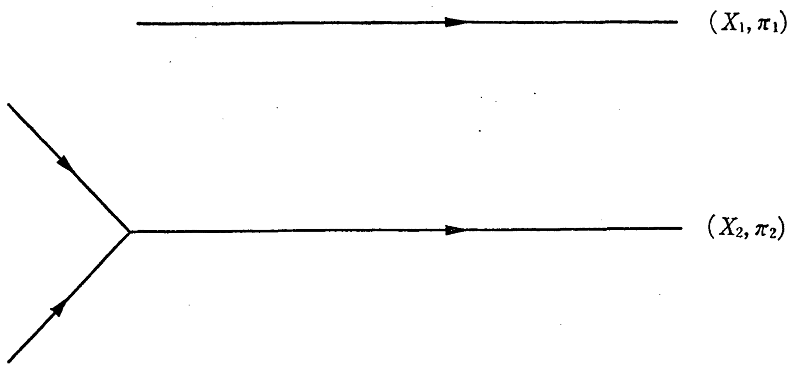
Define $\pi_2: X_2\times R^+\to X_2$ by

$$\pi_2((x_1,x_2),t)=(x_1+t, x'_2),\ (x_1,x_2)\in X_2,\ t\in R^+ \text{ where}$$

$x'_2=\max(x_2-t,0)$ or $\min(x_2+t,0)$ according as $x_2\geqq 0$ or $x_2<0$.

Let (X,π) be the product of (X_1,π_1) and (X_2,π_2). Clearly S^* is (path) connected but S_2 is not.

Theorem 2.4. *Let (X_α,π_α), $\alpha\in a$ and (X,π) be as above. Suppose that S_β is non-empty for unique β in a. Then S^* is (path) connected if and only if both the following conditions hold:*



(a) *S_β is (path) connected.*

(b) *X_α is (path) connected for every $\alpha \neq \beta$.*

Proof. Notice that $S^* = S_\beta \times \prod_{\alpha \neq \beta} X_\alpha$.

Corollary 2.5. *Let (X_α, π_α), $\alpha \in a$ and (X, π) be as above. Let S_β be connected and dense in X_β for some β in a. Let X_α be connected for every $\alpha \neq \beta$. Then S is connected.*

Proof. Here S^* is dense [2, Th. 2.13] in X, etc.,

Theorem 2.6. *Let (X_α, π_α), $\alpha \in a$ and (X, π) be as above. Let there exist an improper start point. The following are equivalent:*

(a) *X is connected.*

(b) *The set of improper start points is connected.*

(c) *The set of start points in (X, π) is connected.*

Proof. (a) implies (b): Let $x = \{x_\alpha\}, y = \{y_\alpha\}$ be improper start points. Let $\tau(x_\alpha)$ be the escape time of x_α. Then there exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in a such that both $\tau(x_{\alpha_n})$ and $\tau(y_{\beta_n})$ converge to zero.

We can take $\alpha_i \neq \beta_j$ for every i and every j. Indeed if $\alpha_i = \beta_j$ for finite number of pairs (i, j), we can drop the terms x_{α_i} and y_{β_j} of the sequences which correspond to such pairs. If $\alpha_i = \beta_j$ for infinite number of pairs (i, j), we can make preliminary adjustments.

Now let $K_1 = \prod_{\alpha = \alpha_n} \{x_\alpha\} \times \prod_{\alpha \neq \alpha_n} (X_\alpha - S_\alpha)$ and $K_2 = \prod_{\alpha = \beta_n} \{y_\alpha\} \times \prod_{\alpha \neq \beta_n} (X_\alpha - S_\alpha)$. Clearly $x \in K_1 \subset S - S^*$ and $y \in K_2 \subset S - S^*$. Since the sets K_1 and K_2 are connected and $K_1 \cap K_2 \neq \phi$, it follows that $K_1 \cup K_2$ is a connected subset of $S - S^*$. Moreover, $x \in K_1$ and $y \in K_2$ etc.,

(b) implies (c): Since $S - S^*$ is dense everywhere [2, Th. 2.12], and $S - S^* \subset S \subset \mathrm{Cl}(S - S^*)$, the result follows.

(c) implies (a): Notice that S is dense [2, Th. 2.12] in X.

Theorem 2.7. *Let (X_α, π_α), $\alpha \in a$ and (X, π) be as above. Let there exist an improper start point. Then the set of improper start points is path connected if and only if X is path connected.*

Proof. Let the set of improper start points be path connected. Let x_β and x'_β be in $X_\beta - S_\beta$ for β fixed. Let $z \in S - S^*$, $z = \{z_\alpha\}$. Pick y and y' in X such that $y_\beta = x_\beta$, $y'_\beta = x'_\beta$ and $y_\alpha = z_\alpha = y'_\alpha$ otherwise. Let $f: I \to S - S^*$ be a path joining improper start points y and y'. Then $\mathrm{Proj}'_\beta \circ f: I \to X_\beta - S_\beta$ is a path joining x_β and x'_β where proj'_β is th erestriction of $\mathrm{proj}_\beta: X \to X_\beta$ to $\mathrm{proj}_\beta^{-1}(X_\beta - S_\beta)$. Thus $X_\beta - S_\beta$ and so, X_β is path connected etc.,

Proof of the converse is left to the reader.

Theorem 2.8. *Let (X_α, π_α), $\alpha \in a$ and (X, π) be as above. Let there exist an improper start point. Then the set S of start points is path connected if and only if X is path connected.*

Proof. Let S^* be non-empty (for otherwise Th. 2.7 applies). Let X be path connected. We need prove that a proper start point and an improper start point can be joined by a path in S. Let $x \in S^*, x = \{x_\alpha\}$ so that $x_\beta \in S_\beta$ for some β in a. Let $y \in S - S^*, y = \{y_\alpha\}$. Let $\tau(y_\alpha)$ be the escape time of y_α. Then there exists a sequence (y_{α_n}) converging to zero. We may take $\beta \neq \alpha_n$ for every n.

Pick $z \in X$ such that $z_\beta = x_\beta, z_\alpha = y_\alpha$ for $\alpha = \alpha_n$ for every n. Since $K = \prod_n \{y_{\alpha_n}\} \times \prod_{\alpha \neq \alpha_n} X_\alpha \subset S$ is path connected and $y, z \in K$, there exists a path $f: I \to S$ joining y to z. Similarly x and z. Hence the result.

Details of the other part are left to the reader.

Corollary 2.9. *Let (X_α, π_α), $\alpha \in a$ and (X, π) be as above. Let there exist an improper start point. The following are equivalent:*

(a) *X is path connected.*

(b) *The set of improper start points is path connected.*

(c) *The set of start points in (X, π) is path connected.*

References

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