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On the Existence of Solutions of Ordinary Differential Equations in Banach Spaces

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We consider the Cauchy problem

$$\text{(CP)} \qquad \frac{dx}{dt}=f(t, x(t)), \quad x(0)=x^0,$$

where $f: N\times V\to E$ is continuous, N is a neighborhood of 0 in the real line, E is an infinite dimensional Banach space with norm $\|\cdot\|$, $V\subset E$ is an open set containing x^0. Then there exist a closed set $Q: \|x-x^0\|\leqq b$ and an interval $[0, a]$ such that f is bounded (i.e. $\|f\|\leqq K$) on $[0, a]\times Q$. Set $\Delta=\min\{a, b/K\}$ and $I=[0, \Delta]$. We are interested in proving the existence of a solution of (CP) defined on I.

Let $C(I)$ be the normed linear space of continuous functions on I with the usual supremum norm (here denoted by $|||\cdot|||$) and $C_Q(I)\subset C(I)$ be the subset of those functions with values in Q. We are going to show the existence of a fixed point of the operator $T: C_Q(I)\to C_Q(I)$ defined by

$$(1) \qquad Tx(t)=x^0+\int_0^t f(s, x(s))ds$$

It is known [5] that such a fixed point would be a (strongly) differentiable function satisfying (CP).

Let us first remark that T is a continuous mapping from $C_Q(I)$ into itself. In fact let x', x_n belong to $C_Q(I)$ and let $x_n\to x'$. Since $f(s, x_n(s))$ converges to $f(s, x'(s))$ pointwise, in view of the extension to integrals of vector-valued functions of the Lebesgue Dominated Convergence Theorem [6], for every $t\in I$, $Tx_n(t)\to Tx'(t)$. The set of functions $\{Tx_n\}$ is equicontinuous and this fact implies [4] that $|||Tx_n-Tx'|||\to 0$.

In what follows we shall make use of the index α introduced by Kuratowski [1]. For the properties of this index we refer to [1], [3] and [8]. We shall repeatedly make use of the following generalization of the Ascoli-Arzelà Theorem due to Ambrosetti [1]:

Theorem 1. *Let $H\subset C(I)$ be bounded and equicontinuous and set $H(t)=\{u=u(t): u(\cdot)\in H\}$. Then*

$$\alpha(H)=\sup\{\alpha(H(t)): t\in I\}.$$

About the existence of solutions of (CP) the following result is known (Szufla [8]):

Theorem 2. *Let* $I=[0, a]$, *E a real Banach space and* $Q=\{x\in E: \|x-x^0\| \leqq b\}$. *If* $f: I\times Q\to E$ *is a continuous mapping such that* $\alpha(f(I\times S))\leqq k\alpha(S)$ *for any subset S of Q, where k is a constant, then* (CP) *admits at least one solution defined on* $[0, h]$ *where* $h\leqq\min(a, b/K)$, $hk<1$ *and* $K=\sup\{\|f(t, x)\|: (t, x)\in I\times Q\}$.

Szufla's theorem strengthenes an analogous result of Ambrosetti in the sense that it requires the continuity of f instead of the *uniform* continuity as in [1].

Let us introduce the function $L: R^+\to R^+$ defined by

$$L(\varepsilon)=\sup\{\alpha(f(I\times S))/\alpha(S)\}$$

the supremum being taken over all subsets $S\subset Q$ such that $\alpha(S)\geqq\varepsilon$. It is clear that $L(\varepsilon)$ increases as ε decreases and it is not difficult to see that $L(\varepsilon)$ is continuous. Moreover we have that

$$\alpha(f(I\times S))\leqq\varepsilon L(\varepsilon) \tag{2}$$

for every $S\subset Q$ such that $\alpha(S)\leqq\varepsilon$. In fact (2) certainly holds if $\alpha(S)=\varepsilon$. Let $\alpha(S)<\varepsilon$, and choose $S^0\subset Q$ such that $\alpha(S^0)=\varepsilon$. Then

$$\alpha(f(I\times S))\leqq\alpha(f(I\times(S\cup S^0)))\leqq\varepsilon L(\varepsilon).$$

A third claim about $L(\varepsilon)$ is that $\varepsilon L(\varepsilon)$ is monotonic non decreasing. In fact assume that there exist ε' and ε'': $\varepsilon'<\varepsilon''$ such that $\varepsilon' L(\varepsilon')=\varepsilon'' L(\varepsilon'')+\xi, \xi>0$. In particular then, $L(\varepsilon'')\leqq L(\varepsilon')-\xi/\varepsilon''$. Set $\eta=\xi/2\varepsilon''$. There exists $S:\alpha(S)\geqq\varepsilon'$ such that $\alpha(f(I\times S))(\alpha(S))^{-1}\geqq L(\varepsilon')-\eta$. Then $\alpha(S)\leqq\varepsilon''$, otherwise $L(\varepsilon'')$ would be not less than $L(\varepsilon')-\eta$. By (2), $\varepsilon'' L(\varepsilon'')\geqq\alpha(f(I\times S))\geqq\alpha(S)(L(\varepsilon')-\eta)\geqq\varepsilon' L(\varepsilon')-\eta\varepsilon''=\varepsilon' L(\varepsilon')-\xi/2$, contradicting the assumption.

We notice that these last properties of L depend on the possibility of choosing subsets of Q having preassigned (sufficiently small) α. In our case this is allowed by the convexity of Q.

Finally let us remark that $L(\varepsilon)$ is $\equiv 0$ when f is a compact mapping and it is bounded by some constant when f is α-lipschitzean. Our theorem 3 below yields existence of solution of (CP) under a condition on $L(\varepsilon)$ that can be considered as an analogue of the Osgood uniqueness condition. In particular, this condition allows $L(\varepsilon)$ to diverge as $\mathrm{Log}(1/\varepsilon)$, as ε tends to zero. More precisely we have

Theorem 3. *Let f be defined and continuous as before. Assume that*

$$\int_{0^+}\frac{d\varepsilon}{\varepsilon L(\varepsilon)}=\infty \tag{3}$$

Then (CP) *admits at least one solution defined on* $[0, \Delta]$.

For its proof we shall need the two following Lemmata. The first Lemma establishes a relation between the α-properties of f and those of the integral operator defined by (1). We use the notation $B[A, \delta]$ for a δ-ball about A and $\overline{co}\{A\}$ for the closed convex hull of A. Moreover when S is a family of mappings, $S(t)$ denotes its *section* at t, i.e. $S(t)=\{x(t): x(\cdot)\in S\}$ and when I is an interval, $S(I)=\{x(t): x(\cdot)\in S \text{ and } t\in I\}$.

Lemma 1. *Let* $I=[0, \Delta]$ *and* $S\subset C_Q(I)$ *be any equicontinuous family of mappings. Set* $\eta=\sup\{\alpha(S(t)): t\in I\}$. *Then*

$$\alpha(T(S))\leqq \Delta L(\eta)\eta.$$

Proof. We are going to show that for every $\xi>0$,

$$\alpha(T(S))\leqq \Delta L(\eta)\eta+\xi$$

Set $\delta'=\xi/4K$. We claim that it is enough to prove that for some partition $0=t_0<t_1<\cdots<t_m=\Delta$ such that $\sup(t_{i+1}-t_i)\leqq\delta'$, for $i=0, 1, \cdots, m$,

$$\alpha(T(S)(t_i))\leqq \Delta L(\eta)\eta+\frac{\xi}{2} \tag{4}$$

In fact if (4) holds, given any $t\in[t_j, t_{j+1}]$,

$$T(S)(t)\subset B[T(S)(t_j), K\delta']=B\left[T(S)(t_j),\ \frac{\xi}{4}\right]$$

i.e.

$$\alpha(T(S)(t))\leqq\alpha(T(S)(t_j))+\frac{\xi}{2}\leqq \Delta L(\eta)\eta+\xi.$$

But then, since the family $T(S)$ is equicontinuous, from Theorem 1 it follows that $\alpha(T(S))\leqq \Delta L(\eta)\eta+\xi$.

Let $\eta'>\eta$ be such that $\eta' L(\eta')\leqq\eta L(\eta)+\xi/3\Delta$. The equicontinuity of S implies that there exists a $\delta''>0$ such that from $t', t''\in I$ and $|t'-t''|\leqq\delta''$ it follows that $\|x(t')-x(t'')\|\leqq(\eta'-\eta)/2$ for every $x\in S$. Let m be a positive integer such that $\Delta/m\leqq\min[\delta', \delta'']$, and take a partition of I into m subintervals of length Δ/m by the points $t_i=i\Delta/m$. Since

$$f(I\times S([t_i, t_{i+1}]))\subset f(I\times B[S(t_i), \frac{1}{2}(\eta'-\eta)])$$

then

$$\alpha(\overline{co}\{f(I\times S([t_i, t_{i+1}]))\})=\alpha(f(I\times S([t_i, t_{i+1}])))\leqq\eta' L(\eta')\leqq\eta L(\eta)+\frac{\xi}{3\Delta}.$$

For each i, cover $\overline{co}\{f(I\times S([t_i, t_{i+1}]))\}$ by a finite number of sets $F_{i,j}$: $j=1, 2, \cdots, \nu(i)$, having diameter not larger than $\eta L(\eta)+\xi/2\Delta$. Consider the $\nu(0)\nu(1)\cdots\nu(m-1)$ subsets of $C(I)$: $C(j(0), j(1), \cdots, j(m-1))$, where $y\in C(j(0), j(1), \cdots, j(m-1))$ if y is picewise linear, $y(0)=x^0$, $y'(t)$ is constant on $t_i<t<t_{i+1}$

and $y' \in F_{i,j(i)}$ there. Each $C(j(0), j(1), \cdots, j(m-1))$ is an equicontinuous family of functions and the diameter of its section at $t=t_i$ is not larger than $i\Delta/m$ $(\eta L(\eta)+\xi/2\Delta) \leqq \Delta\eta L(\eta)+\xi/2$. It is now simple to see that for each i, the section of $T(S)$ at $t=t_i$ is covered by the union of the sections at $t=t_i$ of the $C(j(0), j(1), \cdots, j(m-1))$. In fact let $x \in S$: by the mean value theorem,

$$x^0+\int_0^\Delta f(s, x(s))ds = x^0+\sum_{i=0}^{m-1}\zeta_i\frac{\Delta}{m}$$

where $\zeta_i \in \overline{co}\{f(s, x(s)) : t_i \leqq s \leqq t_{i+1}\}$, i.e. $\zeta_i \in F_{i,j(i)}$ for some choice of the $j(i)$. The continuous and picewise linear function y such that $y(0)=x^0$ and $y'(t) \equiv \zeta_i$ on $t_i < t < t_{i+1}$ belongs to $C(j(0), j(1), \cdots, j(m-1))$ and agrees with $T(x)$ at every t_i. Therefore the section of $T(S)$ at each t_i is covered by the (finite) union of the sections of the $C(j(0), j(1), \cdots, j(m-1))$ so that its α is not larger than $\Delta\eta L(\eta)+\xi/2$. By our previous claim the Lemma is proved.

In particular, when f is α-lipschitzean of modulus k, T is α-lipschitzean (on equicontinuous sets) of modulus $k\Delta$. A result of this kind was established by Ambrosetti [1] under the hypothesis of uniform continuity of f.

The following Lemma was essentially proved in [3].

Lemma 2. *Let $\mathcal{M}$ be a closed, bounded and convex subset of a Banach space X and T: $\mathcal{M} \to \mathcal{M}$ be continuous and such that for some $\varepsilon > 0$ there exists a constant $h < 1$ such that $\alpha(T(S)) \leqq h\alpha(S)$ for every $S \subset \mathcal{M}$ such that $\alpha(S) \geqq \varepsilon$. Then there exists a closed and convex subset $M \subset \mathcal{M}$ such that $T(M) \subset M$ and $\alpha(M) \leqq \varepsilon$. Moreover, M contains the (possibly empty) set of fixed points of T on $\mathcal{M}$.*

Proof of Theorem 3. Roughly speaking, the proof goes as follows: fixing $\eta > 0$ we see that by taking δ sufficiently small (say $\delta \leqq (2L(\eta))^{-1}$), in view of Lemma 1, the operator T considered on $[0, \delta]$ acts as an α-contraction on any equicontinuous subset S of $C[0, \delta]$, such that $\alpha(S) \geqq \eta$. Applying Lemma 2, we infer the existence of a subset of $C[0, \delta]$, mapped into itself by T, whose α is not greater than η. If we wish to apply this process again, in order to have a nested sequence of sets whose α's go to zero, we face the problem that the interval $[0, \delta]$ (in general) tends to zero with η. Since we need to have sets of functions defined on a constant interval, say $[0, \Delta]$, the problem then becomes that of showing the existence of a subset of $C[0, \Delta]$, mapped into itself by T, whose α goes to zero with η. For a fixed η, the construction of such a set is carried by subdividing the inteval $[0, \Delta]$ into a number of sub-intervals and by applying Lemma 2 to each of them.

Since the case $L(\varepsilon) \equiv 0$ has been already considered by Corduneanu [2], for definiteness we shall assume $L(\varepsilon) > 1/2$. Consider the Cauchy problem for the scalar differential equation

(CP$_1$) $$\dot{y}=\frac{yL(y)}{L(\varepsilon)},\quad y(1)=3\varepsilon$$

whose solution satisfies

$$\int_{3\varepsilon}^{y(t)}\frac{dy}{yL(y)}=\frac{t-1}{L(\varepsilon)}.$$

Setting $[\Delta L(\varepsilon)]$=the smallest integer not smaller than $\Delta L(\varepsilon)$, in particular we have

$$\int_{3\varepsilon}^{y(2[\Delta L(\varepsilon)])}\frac{dy}{yL(y)}=\frac{2[\Delta L(\varepsilon)]-1}{L(\varepsilon)}$$

Since the right hand side ramains bounded as ε decreases, condition (3) implies that $y(2[\Delta L(\varepsilon)])\to 0$ as $\varepsilon\to 0$.

Let $\{\eta_n\}$ be a sequence of positive real numbers such that $\eta_n\downarrow 0$. Let $\varepsilon_1<\eta_1/3$ be so small that setting $\varepsilon=\varepsilon_1$ in (CP$_1$), for the corresponding solution we have $y(2[\Delta L(\varepsilon_1)])<1/2\eta_1$. Let $\xi>0$ (but $\xi<\min\{1,\varepsilon_1\}$) be so small that the solution of

(CP$_2$) $$\dot{y}=\frac{yL(y)}{L(\varepsilon_1)}+\xi,\quad y(1)=3\varepsilon_1$$

lies on $[1,2[\Delta L(\varepsilon_1)]]$ within an $(1/2\ \eta_1)$-ball about the previously chosen solution of (CP$_1$).

Set $\delta=\Delta/2[\Delta L(\varepsilon_1)]$. Set also $C_Q^*(I)$ to be the subset of $C_Q(I)$ consisting of those functions that are lipschitzean with Lipschitz constant K. Let $\mathcal{M}_1$ be the closed and convex subset of $C_Q[0,\delta]$ consisting of the restrictions to $[0,\delta]$ of the functions of $C_Q^*(I)$ and call $T_1:\mathcal{M}_1\to\mathcal{M}_1$ the operator T considered as acting on $[0,\delta]$. Let $S\subset\mathcal{M}_1$ be any set such that $\alpha(S)\geqq\varepsilon_1$. Then

$$\alpha(T_1(S))\leqq\delta L(\alpha(S))\alpha(S)\leqq\delta L(\varepsilon_1)\alpha(S)\leqq\frac{1}{2}\alpha(S)$$

and by Lemma 2 there exists a closed and convex subset $M_1\subset\mathcal{M}_1$, mapped into itself by T_1 and such that $\alpha(M_1)\leqq\varepsilon_1$.

We claim that we can define a family $\{M_i\}, i=2,\cdots,\Delta/\delta$, with the following properties:

a) M_i is a closed and convex subset of $C_Q^*[0,i\delta]$, mapped into itself by T_i, where T_i is T considered on $C_Q[0,i\delta]$.

b) Set $\alpha_i=\max\{\alpha(M_i),\varepsilon_1\}$ and $L_i=L(\alpha_i)$. Then

$$\alpha(M_i)-\alpha_{i-1}\leqq 2\delta\alpha_{i-1}L_{i-1}+\xi.$$

In fact, having defined M_i that satisfies a) and b), first set $\mathcal{M}_{i+1}$ to be the subset of $C_Q^*[0,(i+1)\delta]$ consisting of those functions whose restriction to $[0,i\delta]$ belong to M_i. Then $\mathcal{M}_{i+1}$ is closed and convex. Let S be any subset of $\mathcal{M}_{i+1}$ and let us evaluate $\alpha(T(S)(t))$ for $t\in[0,(i+1)\delta]$. Since $T_i(M_i)\subset M_i$, $\sup\{\alpha(T(S)(t)): t\in[0,i\delta]\}\leqq\alpha(M_i)$. For $t\in[i\delta,(i+1)\delta]$, the function $T_{i+1}(x)(t)$ can be rewritten as

$$T_{i+1}(x)(t)=T_i(x)(i\delta)+\int_{i\delta}^t f(s,x(s))ds.$$

Since $\alpha(A+B)\leqq\alpha(A)+\alpha(B)$, applying Lemma 1 and taking into account the monotonicity of $\varepsilon L(\varepsilon)$, we have:

$$\begin{aligned}(5)\qquad \alpha(T_{i+1}(S)(t))&\leqq\alpha(T_i(S)(i\delta))+\delta L(\alpha(S))\alpha(S)\\&\leqq\alpha(M_i)+\delta L(\alpha(S))\alpha(S)\end{aligned}$$

Set $\alpha_i'=\alpha_i(1-\delta L_i)^{-1}$. Then the following identity can be checked:

$$\begin{aligned}&\alpha(S)(\alpha_i'+\delta L_i\xi)(\alpha_i'+\xi)^{-1}\\&=\alpha_i+\delta L_i\alpha(S)+\alpha_i'(\alpha(S)-\alpha_i'-\xi)(1-\delta L_i)(\alpha_i'+\xi)^{-1}.\end{aligned}$$

Let us apply (5) to any $S\subset\mathcal{M}_{i+1}$ such that $\alpha(S)\geqq\alpha_i'+\xi$. Then

$$\alpha(T_{i+1}(S))\leqq\alpha_i+\delta L_i\alpha(S)+\alpha_i'(\alpha(S)-\alpha_i'-\xi)(1-\delta L_i)(\alpha_i'+\xi)^{-1},$$

since the added term in the right hand side is non negative. By the preceding identity,

$$\alpha(T_{i+1}(S))\leqq\alpha(S)((\alpha_i'+\delta L_i\xi)(\alpha_i'+\xi)^{-1})$$

Since ξ is positive and $\delta L_i\leqq 1/2$, the coefficient of $\alpha(S)$ in the above inequality is <1. Therefore, applying Lemma 2 to the operator T_{i+1} and to the set $\mathcal{M}_{i+1}$, we infer the existence of a closed and convex subset of $\mathcal{M}_{i+1}$, say M_{i+1}, mapped into itself by T_{i+1}, and such that

$$\alpha(M_{i+1})\leqq\alpha_i(1-\delta L_i)^{-1}+\xi.$$

This last inequality can be rewritten as

$$\alpha(M_{i+1})-\alpha_i\leqq\delta L_i(1-\delta L_i)^{-1}\alpha_i+\xi$$

and finally

$$\alpha(M_{i+1})-\alpha_i\leqq 2\delta L_i\alpha_i+\xi.$$

Therefore M_{i+1} satisfies both a) and b). We are interested in the properties of of M_n, were $n=\Delta/\delta=2[\Delta L(\varepsilon_1)]$. M_n is a subset of $C_Q(I)$ and we claim that $\alpha(M_n)\leqq\eta_1$. Consider those i such that $\alpha_i=\varepsilon_1$ (1 is one such i) and let j be their maximum. Then either $j\geqq n-1$, so that $\alpha(M_n)\leqq 2\varepsilon_1+\xi<3\varepsilon_1\leqq\eta_1$, or else $j<n-1$. In this is the case, $\varepsilon_1<\alpha(M_{j+1})\leqq 3\varepsilon_1$ and

$$\alpha(M_{s+1})-\alpha(M_s)\leqq 2\delta\alpha(M_s)L(\alpha(M_s))+\xi,\quad s\geqq j+1.$$

Then $\alpha(M_n)$ is dominated by the maximal solution of

$$\alpha(M_{s+1})-\alpha(M_s)\leqq 2\delta\alpha(M_s)L(\alpha(M_s))+\xi,\ \ \alpha(M_{j+1})\leqq 3\varepsilon_1$$

or, with the obvious meaning of the symbols, by the maximal solution of

$$y(s+1)-y(s)\leqq 2\delta y(s)L(y(s))+\xi,\ \ y(j+1)\leqq 3\varepsilon_1.$$

The function y, defined on the integers, can be extended linearly to every interval $[s, s+1]$. Then (by the monotonicity of $yL(y)$), y satisfies the differential inequality

$$D_+y\leqq 2\delta yL(y)+\xi,\ \ y(j+1)\leqq 3\varepsilon_1$$

where D_+ denotes the right derivative. By Theorem 4.1 of [5], it is dominated by the solution of (CP_2) i.e. $\alpha(M_n)\leqq\eta_1$.

Setting $C_Q^{*1}=M_n$, we see that, starting from C_Q^*, we have found a closed and convex subset of C_Q^*, namely C_Q^{*1}, mapped into itself by T, such that $\alpha(C_Q^{*1})\leqq\eta_1$. Since the only properties of C_Q^* used were that C_Q^* is equicontinuous, closed and convex, and mapped into itself by T, and these properties are all shared by C_Q^{*1}, this procedure can be applied again to yield existence of C_Q^{*2}, a subset of C_Q^{*1}, such that $\alpha(C_Q^{*2})\leqq\eta_2$. By iterating we obtain a nested sequence $\{C_Q^{*i}\}$ of closed and convex subsets of $C_Q^*(I)$ such that $\alpha(C_Q^{*i})\leqq\eta_i\downarrow 0$.

It follows then that their intersection is a non-empty compact and convex set mapped into itself by T. As in Darbo [3], the proof of the existence of a fixed point of the continuous mapping T is concluded by applying Schauder's Theorem to this intersection.

Remark. By our proof, every fixed point of T on $C_Q^*(I)$ is contained in the compact set $\cap C_Q^{*i}$. Since every possible solution of (CP) is contained in $C_Q^*(I)$, it is contained in $\cap C_Q^{*i}$. The continuity of T implies that the set of solutions of (CP) is closed and from the above it follows that it is actually compact in $C(I)$. However there are, in the infinite dimensional case, continuous f such that the set of solutions of (CP) is not compact. In l^∞ set $x=(x_1, x_2, \cdots)$ and consider the mapping $f\colon x\to 2(x/\|x\|^{1/2})$, $x\neq 0$ and $f(0)=0$ and the Cauchy problem with $x(0)=0$. Then the functions $x_1(t)=(t^2, 0, 0, \cdots)$, $\cdots, x_n(t)=(0, \cdots, t^2, 0, \cdots)$ are, on any interval $[0, h]$, solutions of the given problem, and it is rather simple to see that such a set of functions is not precompact in the Banach space of continuous function from $[0, h]$ into l^∞. Therefore is does not seem likely that existence of solutions for a mapping f like the above (that does not satisfy the conditions or our Theorem 3) could be proved by a method, like ours, that eventually ends up in applying a fixed point theorem over a compact set.

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