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A Supplement to the Paper "On a Compact Invariant Set Isolated from Minimal Sets"

By Tosiya SAITO

(University of Tokyo and University of Warwick)

1. The following theorem is an easy consequence of Theorem 2 of [1].

Theorem 1. *If F is stable and ∂F contains a point not belonging to any minimal sets, then ∂F contains at least one saddle minimal set.*

Here F denotes, as in [1], a non-empty, compact and non-open invariant set isolated from minimal sets.

As an immediate corollary of the theorem, we get:

Corollary. *If F is positively stable and non-dense, every minimal set in F is either positively stable or a saddle set.*

2. The following theorem gives a slight improvement of Ura's criterion of stability.

Theorem 2. *Let M be a compact invariant set (not necessarily isolated from minimal sets). Then M is positively stable if and only if $D^+(K)\subset M$ for every minimal set K in ∂M.*

Proof. Necessity is obvious since the stability of M implies $D^+(M)=M$ (Ura's criterion). Conversely suppose that $D^+(K)\subset M$ for every minimal set K in ∂M. To prove that M is stable, it is sufficient to show that $x\notin M$ implies $D^-(x)\cap M=\phi$.

Suppose that there exists $x\notin M$ such that $D^-(x)\cap M\neq\phi$. Evidently $D^-(x)\cap M=D^-(x)\cap\partial M$ since $x\notin M$ and M is compact and invariant. Also, as ∂M is compact and invariant, $D^-(x)\cap\partial M=J^-(x)\partial M$ and hence $D^-(x)\cap\partial M$ is also a compact invariant set. Let K be a minimal set in $D^-(x)\cap\partial M$. Then if $y\in K\subset D^-(x)\cap\partial M$, we have $x\in D^+(y)\subset D^+(K)$ which contradicts $D^+(K)\subset M$.

3. Let M be a compact invariant set (not necessarily isolated from minimal sets). Let $S^+(M)$ be the smallest positively stable compact invariant set containing M, and $\sum^+(M)$ the smallest positively asymptotically stable compact invariant set containing M. (Of course, $S^+(M)$ or $\sum^+(M)$ may not exist.)

The following theorem is an immediate consequence of Theorem 2 above and Theorem 5 of [1].

Theorem 3. *If:*

1) *$D^+(M)$ is compact,*

2) *for every minimal set* K *in* $\partial D^+(M), D^+(K) \subset D^+(M)$,
then $S^+(M) = D^+(M)$.

If we further assume that:

3) $D^+(M)$ *is isolated from minimal sets*,
then $\Sigma^+(M) = S^+(M) = D^+(M)$.

Conversely, $S^+(M) = D^+(M)$ *implies* 1) *and* 2), *and* $\Sigma^+(M) = S^+(M) = D^+(M)$ *implies* 1), 2) *and* 3).

Corollary. *If*:

1) $D^+(M)$ *is compact*,

2) $\partial D^+(M) \backslash M$ *contains no minimal sets*,

then $S^+(M) = D^+(M)$.

4. It is well known that if M is a weak attractor, $\Sigma^+(M) = S^+(M) = D^+(M)$ (cf. [2]). Therefore, if M is a weak attractor, it must satisfy 1), 2) and 3) of Theorem 3. The situation will be made clear by the following theorem.

Theorem 4. *A compact invariant set* M *is a positive weak attractor if and only if*:

1) $D^+(M)$ *is compact*,

2) $D^+(M) - M$ *contains no minimal sets*,

3) $D^+(M)$ *is isolated from minimal sets*.

Proof. Suppose that M is a weak attractor. Then it is known that $D^+(M)$ is compact and the region of weak attraction of M:

$$a^+(M) = [x;\ L^+(x) \cap M \neq \phi]$$

is a neighbourhood of $D^+(M)$. So we have only to show that $a^+(M) - M$ contains no minimal sets. But this is obvious from the definition of $a^+(M)$.

Conversely suppose that 1), 2) and 3) are satisfied. Then there exists a relatively compact neighbourhood U of $D^+(M)$ such that $\bar{U} - M$ contains no minimal sets. We shall show that if V is a sufficiently small neighbourhood of M, we have $\pi(x, t) \in U$ for any $x \in V$ and any $t > 0$.

If we assume the contrary, there exists $\{x_n\} \subset U - M$ such that $x_n \to x \in M$ and

$$\pi(x_n, t_n) \in \partial U$$

for some $t_n > 0$. Then, ∂U being compact, $\{\pi(x_n, t_n)\}$ has a cluster point $y \in \partial U$. Then $y \in D^+(x) \subset D^+(M)$ and hence $D^+(M) \cap \partial U \neq \phi$. This is, however, a contradiction since U is a neighbourhood of $D^+(M)$.

Therefore, if V is sufficiently small, we have $\pi(x, t) \in U$ for any $x \in V$ and any $t > 0$. Hence we have $L^+(x) \subset \bar{U}$. But as $\bar{U} - M$ contains no minimal sets by assumption, this implies $L^+(x) \cap M \neq \phi$. Thus we have

$$a^+(M) \supset V$$

and M is a positive weak attractor.

References

[1] T. Saito, On a compact invariant set isolated from minimal sets, Funkcial. Ekvac. **12** (1969), 39–49.

[2] N. P. Bhatia, Weak attractors in dynamical systems, Bol. [Soc. Mat. Mexicana, **11** (1966), 56–64.

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