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On Unstable Invariant Sets

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1. Introduction.

In this paper we study the trajectory structure in a (relatively compact) neighbourhood of a closed set which is invariant and positively unstable under a flow on a metric space. For an understanding of this and the rest of the introduction it is sufficient for the reader to have in mind as an example of an abstract flow a system of first order, autonomous, ordinary differential equations describing mathematically a time-independent physical system; in short, a dynamical system. In a flow a set M is *invariant* if it is a union of trajectories and *positively unstable* if the trajectories through certain points arbitrarily close to M leave a particular neighbourhood of M in the future. Throughout this introduction we consider a closed set M, having a neighbourhood with compact closure and which is invariant and positively unstable under the flow. The instability of such a set is characterized by the existence of certain closed sets which are negatively invariant (i. e. the unions of negative semi-trajectories) and which intersect both M and its complement. These sets are here called *positive prolongation elements from* M. Ura [12] has considered very similar, if larger, sets but he does not study the trajectory structure on them in the same manner as we do here. A preliminary study of prolongation elements, sufficient for later needs, is made in Section 3. This study is deepened in Section 4, which contains the main results of the paper. The section itself falls naturally into two parts, now outlined in turn.

We are able to show in Theorem 1 the existence of a negative semi-trajectory γ^- on any positive prolongation element and in any neighbourhood of M. This result is not entirely new. In a somewhat different, if allied, context Ura and Kimura [13] have proved essentially the same. The analysis continues with a study of the negative limit set L^- of the semi-trajectory γ^-. *The negative limit set* of a trajectory is the set of states (i. e. points of the phase space) approached infinitely often and far into the past. In Theorem 2 it is shown that L^- either is a non-dense minimal set in M or contains a non-dense, minimal, saddle set. In a flow a set is *minimal* if it is closed, invariant and irreducible with respect to these properties. A *saddle set* is a set strongly unstable by virtue of the fact that the trajectories through certain points arbitrarily close to it leave a particular neighbourhood of it both in the past and in the future.

From Theorem 2 it is possible to make deductions (Theorems 3, 4, 5) generalizing results of Ura and Bass as we explain below. Most importantly in our Theorem 4 we show that either some point outside M has its negative limit set intersecting M or there is outside M on any positive prolongation element and in any neighbourhood of M a non-dense, minimal, saddle set.

Unstable invariant sets have been studied in the past by authors, notably Bass, Ura and Zubov, seeking to characterize stability, or equivalently instability. We now wish to outline roughly this work in relation to the first part of Section 4 up to Theorem 5.

The definition of instability is in terms of the flow structure exterior to the set. It is thus natural in seeking to characterize instability to turn one's attention to the flow structure outside the set. In this regard

Zubov's Proposition: *A necessary and sufficient condition for the positive instability of M is that some point outside M should have its negative limit set intersecting M,*

is very appealing through its elegant use of limit sets, whose importance has long been recognised in the study of flows, and because it is known to be true for an *isolated* singular point of a planar dynamical system. See, for example, Nemytskii and Stepanov ([8], 3.42, p. 72). It should be noted that in the literature Zubov's proposition is usually quoted in the form of a criterion (i. e. a necessary and sufficient condition) for stability rather than for instability. In the Russian original of his book ([14], Theorem 7) Zubov asserted the validity of the proposition without restriction on the flow structure outside M. Bass [1] and Mendelson observed, however, that the condition therein is merely sufficient and not necessary for the positive instability of M. Ura ([12], § 3, p. 257) has given counter-examples, including one of a non-isolated singular point of a planar dynamical system and one of an isolated singular point of a dynamical system in Euclidean 3-space.

The work of Bass [1] and much of the work of Ura in [12] have been devoted to correcting Zubov's proposition. This entails considering positively unstable sets for which Zubov's criterion fails and the analysis is best viewed as showing that sets are of this type only when there is a particular flow structure outside them. Bass' analysis produces an instability criterion ([1], Theorem 2) which Ura ([12], Theorem 2 (10′), p. 284) has generalized. The details are too complicated to give here. Besides the present author believes that Bass' proof fails (thus jeopardising Ura's generalization, although there are no details of proof). In our Theorem 5 we give a correction which strengthens Bass' result in certain respects while weakening it in another. Ura's analysis ([12], Theorem 2, p. 284) is more extensive and it is in fact possible to deduce our

Theorem 4 from it.

The motivation of the present paper was to settle the

Conjecture: *Either there is a point outside M with its negative limit set contained in M or there is a minimal set outside of but in any neighbourhood of M.* The second part of Section 4 from Theorem 6 onwards, reports the progress made towards proving this conjecture, in the absence of a counter-example. In Theorem 6 the flow on that part of the limit set L^- outside M is analysed using a result of Friedlander ([4], Theorem 9) but the details are too complicated to give here. Certain deductions (Theorems 7, 8, 9) are then possible; most notably Theorem 8 in which we replace the second alternative of the conjecture by the following: there is on any positive prolongation element, in any neighbourhood of but not wholly contained in M a non-dense, negative limit set which is also a saddle set. This considerably strengthens the only other result in this direction to date, viz, that of Ura ([12], Theorem 3, p. 286), which shows that if there is no point outside M with its negative limit set contained in M there exists a closed, invariant set in any neighbourhood of but not wholly contained in M.

In Section 5 we prove the conjecture proposed above for continuous flows on a class of 2-manifolds which includes the Euclidean plane $\boldsymbol{R}^2$. This is possible because there is at one's disposal for these flows a theory generalising the classical results of Poincaré and Bendixson in the qualitative study of planar dynamical systems. For this theory we rely upon Hájek's recent book [6]. The phase-spaces under consideration in this section are separated by Jordan curves (i. e. homeomorphs of the 1-sphere $\boldsymbol{S}^1$) in the same manner as the Euclidean plane is. Since the Poincaré-Bendixson theory is unlikely to generalize to flows on spaces without this property the method here employed would appear unfruitful for resolving the conjecture in general.

In the final section of the paper we digress from the main theme in order to point out that there is a *Lagrange stable* semi-trajectory (i. e. a semi-trajectory with compact closure) outside of but in any neighbourhood of a closed, non-open, invariant set having a neighbourhood with compact closure. This is doubtless well known but to the author's knowledge does not appear explicitly in the literature except in the one special case of a singular point of a planar dynamical system. See Sansone and Conti ([10], Theorem 26, p. 178). Indeed Mendelson [7] has given a sufficient condition for the existence of such a semi-trajectory near an isolated singular point of a dynamical system in Euclidean n-space $\boldsymbol{R}^n$. This condition is here proved superfluous.

We now turn to giving in the next section precise basic definitions and notations.

2. Basic definitions and notations.

In what follows X is a metric space with metric ρ. If $M\subseteq X$ and $N\subseteq X$, $\bar{M}$ dentoes the closure of M, ∂M the boundary of M and $N\backslash M$ the complement of M in N. If $M\subseteq X$ and $\varepsilon>0$, $S(M,\varepsilon)$ denotes the set $\{x\in X: \rho(x,M)<\varepsilon\}$ and $S[M,\varepsilon]$ its closure. If $M\subseteq X$ and, for each $x\in X$, $L(x)$ is a subset of X then $L(M)$ denotes the set $\cup\{L(x): x\in M\}$.

Let $\mathcal{T}$ denote either the additive group $\boldsymbol{R}$ of real numbers with the usual topology or the additive group $\boldsymbol{Z}$ of integers with the discrete topology and let π be a continuous map from $X\times\mathcal{T}$ into X such that for each $x\in X$ and each $s,t\in\mathcal{T}$

$$\pi(x,0)=x,\quad \pi(\pi(x,s),t)=\pi(x,s+t).$$

Then $(X,\mathcal{T},\pi)$ is called a *flow* on X and is said to be *continuous* or *discrete* according as $\mathcal{T}$ is $\boldsymbol{R}$ or $\boldsymbol{Z}$. For $x\in X$ and $t\in\mathcal{T}$ it is more convenient to write xt instead of $\pi(x,t)$ if no confusion can arise. The so-called "identity" and "homomorphism" properties of π may then be expressed:

$$x0=x,\quad (xs)t=x(s+t)$$

for each $x\in X$ and each $s,t\in\mathcal{T}$.

Continuous flows arise naturally in the study of time-independent physical systems which can be described in terms of a system of first order, autonomous, ordinary differential equations satisfying certain conditions. Discrete flows determine and are determined by homeomorphisms of X onto itself. Apropos these remarks see, for example, Gottschalk ([5], pp. 337, 338).

An important property of flows is the so-called *continuity with respect to initial position*: for any $x\in X$, $\tau>0$ and $\varepsilon>0$ there exists $\delta>0$ such that $\rho(xt,yt)<\varepsilon$ for all y with $\rho(x,y)<\delta$ and all t with $|t|\leqq\tau$. For a proof in the continuous case see, for example, Nemytskii and Stepanov ([8], 2.01, p. 327).

Suppose that as regards the flow $(X,\mathcal{T},\pi)$ a meaning has been assigned, arising from the ordering of $\mathcal{T}$, to the locution "* is a positive **" (or "* is positively **"). Then we use the locution "* is a negative **" to mean that * is a positive ** under the "time-reversed" flow $(X,\mathcal{T},\mu)$ where $\mu(x,t)=\pi(x,-t)$ for each $x\in X$ and each $t\in\mathcal{T}$. Definitions and enunciation and proofs of theorems are uaually given only in the positive form. Use is made as required, however, of oppositely sensed variants. Notations are sensed by affixing to them the appropriate $+$ or $-$ sign.

For $x\in X$ the set $\gamma(x)=\{xt\in X: t\in\mathcal{T}\}$ is called the *trajectory of* x and the set $\gamma^+(x)=\{xt\in\gamma(x): t\geqq0\}$ is called the *positive semi-trajectory of* x. A semi-trajectory (either positive or negative) is said to be *Lagrange stable* if it has compact closure. For $x \in X$ and $t_1, t_2 \in \mathcal{T}$ with $t_1 < t_2$, the set $\{xt \in \gamma(x) : t_1 \leqq t \leqq t_2\}$ is called the *trajectory arc of* $\gamma(x)$ with *end points* xt_1 and xt_2.

A non-void set $M \subseteq X$ is said to be *invariant* if $\gamma(M) = M$ and *positively invariant* if $\gamma^+(M) = M$. A closed, invariant set is said to be *minimal* if it has no proper, closed, invariant subset. A point $x \in X$ is said to be a *fixed (critical, singular, rest* or *equilibrium) point* if $\{x\}$ is invariant. A trajectory is called a *cycle* if, topologically, it is a Jordan curve i.e. if it is homeomorphic to the 1-sphere S^1. Clearly both fixed points and cycles are minimal.

For $x \in X$ the *positive* (or *omega*) *limit set of* x, denoted $L^+(x)$, is the set of all points $y \in X$ for which there exists a sequence $\{t_n\}$ in $\mathcal{T}$ with $t_n \to +\infty$ and $xt_n \to y$.

For $x \in X$ the *positive prolongation of* x, denoted $D^+(x)$, is the set of all points $y \in X$ for which there exist sequences $\{x_n\}$ in X and $\{t_n\}$ in $\mathcal{T}$ with $x_n \to x$, $t_n \geqq 0$ for all n and $x_n t_n \to y$.

For $x \in X$ the *positive prolongational limit set of* x, denoted $J^+(x)$, is the set of all points $y \in X$ for which there exist sequences $\{x_n\}$ in X and $\{t_n\}$ in $\mathcal{T}$ with $x_n \to x$, $t_n \to +\infty$ and $x_n t_n \to y$.

A non-void set $M \subseteq X$ is said to be *positively stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\gamma^+(S(M, \delta)) \subseteq S(M, \varepsilon)$. A set M may be trivially stable by virtue of the fact that there are no points of $X \backslash M$ arbitrarily close to M. If the set M is closed and has a neighbourhood with compact closure, it is trivially stable iff it is open. A non-void set $M \subseteq X$ is said to be *positively unstable* if it is not positively stable.

Let $\mathcal{S}$ be a set of trajectory arcs and let $M \subseteq X$. Then M is called a *saddle set with respect to* $\mathcal{S}$ if there is a neighbourhood U of M such that any neighbourhood V of M intersects some member of $\mathcal{S}$ with both end points outside U. A set $M \subseteq X$ is called a *saddle set* if it is a saddle set with respect to some set of trajectory arcs. A set $M \subseteq X$ is a saddle set iff there exists a neighbourhood U of M such that any neighbourhood V of M contains a point x such that $\gamma^+(x) \not\subseteq U$ and $\gamma^-(x) \not\subseteq U$. This is in fact the definition of saddle set adopted by Ura ([12], Definition 1, p. 277), who attributes the concept of P. Seibert. Saddle sets are clearly unstable in both senses. When a positive limit set is also a saddle set we shall call it a *positive saddle limit set*. A fixed point which is also a saddle set shall be called a *saddle fixed point*.

General accounts of continuous flows on metric spaces have been given by Bhatia and Szegö ([2], Chapter 2) and Nemytskii and Stepanov ([8], Chapter 5). The reader is referred to these texts, particularly the former, for further details about most of the concepts defined in this section.

3. Prolongation elements.

Let $x, y \in X$ and $E \subseteq X$ be non-void. Then E is called a *positive prolongation element from x to y* and denoted $D^+(x, y)$ if

(i) there exist a sequence $\{x_n\}$ of points of X and a positive sequence $\{t_n\}$ such that $x_n \to x$, $x_n t_n \to y$, and

(ii) $e \in E$ iff there exist a non-negative sequence $\{\tau_n\}$ with $0 \leqq \tau_n \leqq t_n$ for all n and a subsequence of $\{x_n \tau_n\}$ tending to e.

The sequence $\{\Gamma_n\}$ of trajectory arcs, where for each n, $\Gamma_n = \{x_n t : 0 \leqq t \leqq t_n\}$, will be said to be *associated with E*.

E is called a *positive prolongation element* if it is a positive prolongation element from x to y for some $x, y \in X$.

The concept of prolongation element is closely related to, but is not identical with, that of prolongation with respect to a sequence of points in X and a sequence of intervals in $\mathcal{T}$, as introduced by Ura ([12], Definition 1, p. 264 and Example 1, p. 265). In Ura's terminology the positive prolongation element $D^+(x, y)$ is the (first) prolongation of x with respect to $\{x_n\}$ and the sequence of intervals $\{[0, t_n]\}$. On the other hand, as simple examples show, a prolongation of x with respect to a sequence $\{x_n\}$ of points in X and a sequence $\{[0, t_n]\}$ of intervals in $\mathcal{T}$ is not necessarily a positive prolongation element from x, although it will always contain one.

In the propositions to follow we establish some properties of prolongation elements sufficient for our needs in the next section.

Proposition 1. *Let $x, y \in X$. Then there exists a positive prolongation element from x to y iff $y \in D^+(x)$.*

Remark. Even when a positive prolongation element from x to y exists it need not be unique.

Proposition 2. *Let $D^+(x, y)$ be a positive prolongation element from x to y. Then*

(i) $\{x, y\} \subseteq D^+(x, y) \subseteq D^+(x)$;

(ii) *$D^+(x, y)$ is closed.*

The proofs of both parts are obvious.

Proposition 3. *Let the flow on X be continuous. Suppose $D^+(x, y)$ is a positive prolongation element from x to y which, together with a sequence of associated trajectory arcs, is contained in some compact subset of X. Then $D^+(x, y)$ is connected.*

Since we do not use this proposition in the sequel we omit a proof. A proof exactly similar to that establishing the connectedness of the limit set of a Lagrange stable semi-trajectory is possible. See, for example, Nemytskii and Stepanov ([8], 3.09, p. 342). For a similar result, but differing in context, see Ura ([12], Theorem 1, p. 265).

Proposition 4. *Let $D^+(x, y)$ be a positive prolongation element from x to y and let M be a closed, invariant subset of $D^+(x, y)$. Then*

(i) *if $x \notin M$ and $y \notin M$, M is a saddle set with respect to any sequence of arcs associated with $D^+(x, y)$;*

(ii) *if $x \notin M$ or $y \notin M$, M is non-dense;*

(iii) *if $x \notin M$, $M \subseteq J^+(x)$ and if $y \notin M$, $M \subseteq J^-(y)$.*

Remark on (i). That M is a saddle set if $x \notin M$ and $y \notin M$ follows from a result of Ura ([12], Theorem 1, p. 283) and remarks made above about $D^+(x, y)$. Ura establishes his result in a somewhat different context but his method of proof, suitably transcribed, would apply equally well here.

Proof of (i) and (ii). Let $\{\Gamma_n\}$ be a sequence of trajectory arcs associated with $D^+(x, y)$, where for each n, $\Gamma_n = \{x_n t : 0 \leqq t \leqq t_n\}$; so that $x_n \to x$ and $x_n t_n \to y$.

(i) Suppose that $x \notin M$ and $y \notin M$. Let $2\Delta = \min(\rho(x, M), \rho(y, M))$ so that $\Delta > 0$. Then for all large n, $\min(\rho(x_n, x), \rho(x_n t_n, y)) < \Delta$ so that $x_n \notin S(M, \Delta)$ and $x_n t_n \notin S(M, \Delta)$. Let V be a neighbourhood of M. For arbitrarily large k, since $M \subseteq D^+(x, y)$, $\Gamma_k \cap V \neq \phi$ and from above $x_k \notin S(M, \Delta)$, $x_k t_k \notin S(M, \Delta)$. Thus M is a saddle set with respect to $\{\Gamma_n\}$.

(ii) Suppose that $x \notin M$; an exactly similar proof applies if $y \notin M$. Suppose if possible that $m \in M$ is an interior point with neighbourhood $U \subseteq M$. Then there exists a positive sequence $\{\tau_n\}$ such that $x_n \tau_n \to m$, as we may suppose for notational convenience. Thus for all sufficiently large n, $x_n \tau_n \in U \subseteq M$ and so $x_n \in M$. Since $\rho(x, M) > 0$, $x_n \not\to x$; a contradiction in view of which M has empty interior and so, being closed, is non-dense.

The proof of (iii) is preceded by two lemmas.

Let $Y \subseteq X$ and $y \in Y$. We say that y *leaves Y positively* if $\gamma^+(y) \not\subseteq Y$.

Lemma 1. *If $Y \subseteq X$ is open, $y \in Y$ and y leaves Y positively then there exists $\tau > 0$ such that $y\tau \notin Y$ while $yt \in Y$ for $0 \leqq t < \tau$.*

For a proof define $\tau = \inf\{t > 0 : yt \notin Y\}$. We omit the routine details.

The point $y\tau$ whose existence is guaranteed under the conditions of the previous lemma is called the *first exit-point from Y of $\gamma^+(y)$.*

Lemma 2. *Let $M \subseteq X$ be invariant and $\varepsilon > 0$. Let $\{m_n\}$ be a sequence of points in $X \backslash M$ converging to the point $m \in M$ and such that for all n, m_n leaves $S(M, \varepsilon)$ positively. Then we may assume, by considering a subsequence if necessary, that*

$$1 < \tau_1 < \tau_2 < \cdots < \tau_n < \tau_{n+1} < \cdots \to +\infty,$$

where $m_n\tau_n$ denotes the first exit-point from $S(M,\varepsilon)$ of $\gamma^+(m_n)$.

An inductive proof follows from the continuous dependence on initial position which implies that points of the sequence $\{m_n\}$ sufficientiy close to m will follow it arbitrarily closely and so remain arbitrarily near M for arbitrarily long time. We omit the routine details.

Proof of Proposition 4. (iii). We prove that $M\subseteq J^-(y)$ if $y\notin M$; an exactly similar proof shows that $M\subseteq J^+(x)$ if $x\notin M$.

Let $2\Delta=\rho(y,M)$ so that $\Delta>0$. Let $m\in M\subseteq E$ so that there exists t_n^* with $0\leqq t_n^*<t_n$ for all n such that, as we may suppose, $x_nt_n^*\to m$. For each n, write $m_n=x_nt_n^*$ and $T_n=t_n-t_n^*>0$. We may suppose that for all n, $m_n\in S(m,\Delta)\subseteq S(M,\Delta)$ and $m_nT_n=x_nt_n\in S(y,\Delta)$. Hence m_n leaves $S(M,\Delta)$ positively. If $m_n\tau_n$ denotes the first exit point from $S(M,\Delta)$ of $\gamma^+(m_n)$ we may assume by Lemma 2 that

$$1<\tau_1<\tau_2<\cdots<\tau_n<\tau_{n+1}<\cdots\to+\infty.$$

Since for all n, $T_n>\tau_n$ it follows that $T_n\to+\infty$. Hence $y\in J^+(m)$ and so $m\in J^-(y)$. Thus $M\subseteq J^-(y)$.

Proposition 5. *Let $M\subseteq X$ be a closed invariant set, $x\in M$, $y\in X\backslash M$ and $D^+(x,y)$ be a positive prolongation element from x to y. Then*

(i) $D^+(x,y)\backslash M\subseteq J^+(x)$;

(ii) $D^+(x,y)\backslash M$ *is negatively invariant.*

Proof. (i) The proof follows along much the same lines as that of Proposition 4 (iii). We omit the routine details.

(ii) Let $p\in D^+(x,y)\backslash M$ and $t\leqq 0$. Firstly $pt\notin M$ since $p\notin M$. Since $p\in D^+(x,y)$ there exist sequences $\{x_n\}$ and $\{t_n\}$ such that $t_n>0$ for all n, $x_n\to x$ and $x_nt_n\to y$ and, as we may suppose, a sequence $\{\tau_n\}$ with $0\leqq\tau_n\leqq t_n$ for all n and $x_n\tau_n\to p$. As previously it is no restriction to assume that $\tau_n\to+\infty$. Now for all sufficiently large n, $0\leqq\tau_n+t\leqq t_n$. Further $x_n(\tau_n+t)=(x_n\tau_n)t\to pt$ so that $pt\in D^+(x,y)$. Hence $pt\in D^+(x,y)\backslash M$. Thus $\gamma^-(p)\subseteq D^+(x,y)\backslash M$ and $D^+(x,y)\backslash M$ is negatively invariant.

4. Unstable invariant sets.

Let $M\subseteq X$ and $E\subseteq X$. Then E is called a *positive prolongation element from* M if it is a positive prolongation element from x to y for some $x\in M$, $y\in X\backslash M$.

Throughout the remainder of this section we adopt the following

Standing hypothesis. *M shall denote a closed subset of X, having a neighbourhood with compact closure and invariant under the flow on X.*

Proposition 6. *There exists a positive prolongation element from M iff M is positively unstable.*

This proposition is an immediate consequence of Ura's stability criterion ([11], Theorem 5, p. 177), viz, *M is positively stable iff $D^+(M)=M$.* For a proof of this criterion valid under present hypotheses see Bhatia and Szegö ([2], Theorems 2.6.5, 2.6.6, p. 135).

Theorem 1. *Let M be positively unstable so that positive prolongation elements from M exist. For any positive prolongation element E from M and any neighbourhood U of M there exists a point $p\in(E\cap U)\backslash M$ such that $\overline{\gamma^-(p)}\subseteq E\cap U$.*

Proof. Let U be a neighbourhood of M. Then it is possible to choose $\varepsilon>0$ small enough to ensure that $S[M,\varepsilon]\subseteq U$ and $S[M,\varepsilon]$ is compact.

Let E be a positive prolongation element from M. Thus there exist $m\in M$, $q\in X\backslash M$, a sequence $\{m_n\}$ of points of X and a positive sequence $\{t_n\}$ such that $m_n\to m$ and $m_nt_n\to q$. Choose $\varepsilon>0$ so that, in addition to restrictions already made, $2\varepsilon<\rho(q, M)$. Thus we may suppose that, for all n, $m_n\in S(M, \varepsilon)$ and m_n leaves $S(M,\varepsilon)$ positively. Denote by $m_n(\tau_n+1)$ the first exit point from $S(M,\varepsilon)$ of $\gamma^+(m_n)$. By Lemma 2 we may assume that

$$0<\tau_1<\tau_2<\cdots<\tau_n<\tau_{n+1}<\cdots\to+\infty.$$

By definition of τ_n, $m_n\tau_n\in S(M,\varepsilon)$ so that we may assume, by choosing a subsequence if necessary, that $\{m_n\tau_n\}$ converges, to p say. It is clear that $p\in S[M,\varepsilon]\subseteq U$. Since $m_n(\tau_n+1)\to p1$ and $m_n(\tau_n+1)\notin S(M,\varepsilon)$ for all n, it follows that $p1\notin S(M,\varepsilon)$. Hence $p\notin M$. Since $\tau_n>0$ for all n it is clear that $p\in E$. Thus $\gamma^-(p)\subseteq E$ and $\overline{\gamma^-(p)}\subseteq\bar{E}=E$, E being negatively invariant and closed.

Fix $t\leqq 0$. Now for all sufficiently large n, $0\leqq\tau_n+t\leqq\tau_n$ so that $m_n(\tau_n+t)\in S(M,\varepsilon)$ and

$$\begin{aligned}\rho(pt, M)&\leqq\rho(pt, m_n(\tau_n+t))+\rho(m_n(\tau_n+t), M)\\&<\rho(pt, m_n(\tau_n+t))+\varepsilon.\end{aligned}$$

Let $n\to\infty$ so that $m_n(\tau_n+t)\to pt$ to conclude that $\rho(pt, M)\leqq\varepsilon$. Hence $\gamma^-(p)\subseteq S[M,\varepsilon]$ and $\overline{\gamma^-(p)}\subseteq S[M,\varepsilon]\subseteq U$.

Thus the theorem is proved.

Remark. Note that $q\notin L^-(p)$, by choice of ε and the fact that $L^-(p)\subseteq\overline{\gamma^-(p)}\subseteq S[M, \varepsilon]$. Thus by Proposition 4 (ii), (iii), $L^-(p)$ is non-dense and $L^-(p)\subseteq J^-(q)$.

In a different context Ura and Kimura have shown, *en passant* in Proposition 3 of [13], the existence of a negative semi-trajectory in any neighbourhood of a positively unstable invariant set. When the flow is continuous the present theorem is a special case of their result.

The analysis now continues with a study of the negative limit set $L^-(p)$.

Theorem 2. *Refer to Theorem* 1. *As regards the point* p, $L^{-}(p)$ *either is a non-dense, minimal set in* M *or contains a non-dense, minimal, saddle set.*

Remark. Examples show that these alternatives are not exclusive and neither is superfluous.

The following sufficient condition for a set to be a saddle set will be used in the proof of Theorem 2.

Lemma 3. *The set* $M \subseteq X$ *is a saddle set if*

(i) *there exists* $x \notin M$ *such that* $L^{+}(x) \cap M \neq \phi$ *and* $J^{+}(x) \backslash M \neq \phi$;

or (ii) *there exists* $y \notin M$ *such that* $L^{-}(y) \cap M \neq \phi$ *and* $J^{-}(y) \backslash M \neq \phi$.

The sufficiency of either condition, and the necessity of both, have been proved by Saito ([9], Theorem 2) when the flow is continuous under the additional hypothesis that in some neighbourhood of, but outside M there are no minimal sets. This hypothesis is, however, superfluous to the sufficiency, as his proof shows. Saito also assumes that X is locally compact and M is compact (conditions which together imply our standing hypothesis) although his proof is valid under our standing hypothesis.

Proof of Theorem 2. There are three cases to consider.

(i) If $L^{-}(p)$ is minimal and $L^{-}(p) \cap M \neq \phi$ then $L^{-}(p) \subseteq M$.

(ii) If $L^{-}(p)$ is minimal and $L^{-}(p) \cap M = \phi$ then $m \notin L^{-}(p)$, $q \notin L^{-}(p)$ and it follows from Proposition 4(i) that $L^{-}(p)$ is a saddle set.

(iii) If $L^{-}(p)$ is not minimal consider the minimal sets in $L^{-}(p)$. There is at least one, K say. Further $p \notin K$; for otherwise $L^{-}(p) = K$ and $L^{-}(p)$ is minimal contrary to hypothesis. Clearly $\phi \neq K = L^{-}(p) \cap K$ and $\phi \neq L^{-}(p) \backslash K \subseteq J^{-}(p) \backslash K$ so that, by Lemma 3, K is a saddle set.

Finally minimal sets in $L^{-}(p)$ are necessarily non-dense since $L^{-}(p)$ is. Thus considering the three cases the theorem is proved.

The next theorem, a deduction from Theorems 1 and 2 relevant to a result of Ura, shows the necessity of Zubov's condition for the positive instability of invariant sets with a certain "isolation" property.

Theorem 3. *Let* M *be positively unstable. Suppose there is a positive prolongation element* E *from* M *and a neighbourhood* V *of* M *such that* $(E \cap V) \backslash M$ *contains no non-dense, minimal, saddle set. Then for any neighbourhood* U *of* M *there exists a point* $p \in (E \cap U) \backslash M$ *such that* $\overline{\gamma^{-}(p)} \subseteq U$ *and* $L^{-}(p) \cap M \neq \phi$.

Proof. Let U be a neighbourhood of M. Then there exists a neighbourhood W of M such that $W \subseteq U \cap V$. By Theorem 1 there exists $p \in (E \cap W) \backslash M \subseteq (E \cap U) \backslash M$ such that $\overline{\gamma^{-}(p)} \subseteq E \cap W \subseteq E \cap U \cap V$. Hence $\overline{\gamma^{-}(p)} \subseteq U$ and since $\overline{\gamma^{-}(p)} \subseteq E \cap V$, $L^{-}(p) \subseteq E \cap V$. By Theorem 2, $L^{-}(p)$ either is a minimal set in M, in which case the theorem is proved, or it contains a non-dense, minimal, saddle set, K say. It follows from the hypothesis since $K \subseteq L^-(p) \subseteq E \cap V$ that $K \subseteq M$ and the theorem is proved in this case also.

This theorem clearly generalizes and strengthens, for continuous flows, a result of Ura ([12], Corollary, p. 286), (or rather that part of it which is not an *a fortiori* consequence of the sufficiency of Zubov's condition for instability), viz, *if M is positively unstable and if there are in some neighbourhood of M no non-dense, saddle sets disjoint from M, then $L^-(X \setminus M) \cap M \neq \phi$.*

There is an important if easy consequence of Theorem 3.

Theorem 4. *Let M be positively unstable so that positive prolongation elements from M exist. Then either $L^-(X \setminus M) \cap M \neq \phi$ or for any positive prolongation element E from M and any neighbourhood V of M there is a non-dense, minimal, saddle set in $(E \cap V) \setminus M$.*

Proof. Either $L^-(X \setminus M) \cap M \neq \phi$ or else $L^-(X \setminus M) \cap M = \phi$, in which case the result follows from the contraposition of Theorem 3.

Remarks. 1. Examples show that these alternatives are not mutually exclusive and neither is superfluous.

2. On the other hand if M is positively stable there are no positive prolongation elements from M and, as is well known, $L^-(X \setminus M) \cap M = \phi$.

3. In the second alternative the non-dense, minimal sets in $(E \cap V) \setminus M$ are, in view of Proposition 4 (i), saddle sets with respect to any sequence of trajectory arcs associated with E.

Theorem 4 may be deduced for continuous flows from a result of Ura ([12], Theorem 2, p. 284) if we observe that his analysis, given on a particular prolongation of some point of M with respect to a sequence in X and a sequence of intervals in $\mathcal{T}$, applies equally well on any such prolongation and in particular on any positive prolongation element from M. Thus if $L^-(X \setminus M) \cap M = \phi$, on any positive prolongation element from M, by (7), there is in any neighbourhood of and outside M a minimal set, which by $(10)^+$ is a saddle set, necessarily non-dense by (6). (Figures in parentheses refer to the sections of Ura's theorem.)

Remark on Ura's work. As regards the context of Ura's work in relation to our own we remark here, although it is appropriate elsewhere, that a continuous flow on X will constitute a "uniform dynamical system" in the sense of Ura ([12], §1, p. 252) iff X can be embedded in a space $\tilde{X}$ in such a way that it is open and relatively compact in $\tilde{X}$. See [12], Example 1, p. 254. On the other hand a continuous flow on X restricted in an obvious sense to an open neighbourhood $S(M, \varepsilon)$, where $\varepsilon > 0$ is so small that $S[M, \varepsilon]$ is compact (refer to the standing hypothesis), constitutes a uniform dynamical system in Ura's sense and his analysis applies to this "restricted" flow.

We continue the analysis of the flow near positively unstable sets M for which $L^{-}(X\backslash M)\cap M=\phi$ in the direction of some work of Bass and Ura.

Theorem 5. *Let M be positively unstable and $L^{-}(X\backslash M)\cap M=\phi$. Then for any positive prolongation element E from M there is in M a limit point of a sequence $\{p_k\}$ of points in $E\backslash M$ such that*

1) *for each k, p_k belongs to some minimal set $M_k\subseteq E\backslash M$ and $M_j\cap M_k=\phi$ when $j\neq k$,*

2) *on any sequence of trajectory arcs associated with E there exists a double sequence $\{p_{kn}\}$ of points in $X\backslash M$ such that*

(i) *$p_{1n}\to p_1$ as $n\to\infty$,*

(ii) *for each $n, k, p_{(k+1)n}=p_{kn}\tau_{kn}$ for some $\tau_{kn}\leqq 0$,*

(iii) *for each k, some subsequence of $\{p_{(k+1)n}\}$ tends to p_{k+1} and if, for notational convenience, we assume this subsequence to be $\{p_{(k+1)n}\}$ then $\tau_{kn}\to-\infty$ as $n\to\infty$.*

Proof. Let E be a positive prolongation element from $m\in M$ to $q\in X\backslash M$ with an associated sequence $\{\Gamma_n\}$ of trajectory arcs. Thus there exists a sequence $\{m_n\}$ with $m_n\in\Gamma_n$ for all n and $m_n\to m$. Let ε_1 be such that $0<\varepsilon_1<1$. By Theorem 4 there exists in $S(M, \varepsilon_1)$ a minimal set $M_1\subseteq E\backslash M$, Let $p_1\in M_1\subseteq E\backslash M$. Thus there exists a positive sequence $\{t_{1n}\}$ such that, as we may suppose, $m_n t_{1n}\to p_1$. Write $p_{1n}=m_n t_{1n}$.

Consider now the positive prolongation element $E_1\subseteq E$ from m to p_1 with associated trajectory arcs $\{\Gamma_{1n}\}$ where $\Gamma_{1n}=\{m_n t: 0\leqq t\leqq t_{1n}\}\subseteq\Gamma_n$. Let ε_2 be such that $0<\varepsilon_2<\min\left(\frac{1}{2}, \rho(M_1, M)\right)$. By Theorem 4 there exists in $S(M, \varepsilon_2)$ a minimal set $M_2\subseteq E_1\backslash M\subseteq E\backslash M$. Clearly $M_1\cap M_2=\phi$. Let $p_2\in M_2\subseteq E_1\backslash M\subseteq E\backslash M$. Thus there exist a positive sequence $\{t_{2n}\}$ with $0<t_{2n}\leqq t_{1n}$ for each n and a subsequence of $\{m_n t_{2n}\}$ converging to p_2. Define $p_{2n}=m_n t_{2n}$, $\tau_{1n}=t_{2n}-t_{1n}$, so that $\tau_{1n}\leqq 0$ and $p_{1n}\tau_{1n}=m_n t_{1n}(t_{2n}-t_{1n})=m_n t_{2n}=p_{2n}$ for all n. Now if for notational convenience we assume that $p_{2n}\to p_2$ by a familiar argument based on Lemma 2 applied to M_1 and $\{\Gamma_{1n}\}$ we may assume that $\tau_{1n}\to-\infty$.

The proof now proceeds iteratively. Repeat the argument with $M_2, p_2, \{p_{2n}\}$ replacing $M_1, p_1, \{p_{1n}\}$ respectively in order to determine M_3, p_3 and $\{p_{3n}\}$ etc. At the k-th stage of the proof are determined a minimal set M_k in $(S(M, 1/k)\cap E)\backslash M$ disjoint from $M_1, M_2, \cdots, M_{k-1}$, a point $p_k\in M_k$ and a sequence $\{p_{kn}\}$ with the required properties. We omit the routine details.

For continuous flows, Bass ([1], Theorem 2) has given a stability criterion. The non-trivial part of his result may be stated thus:

If M is positively unstable and $L^{-}(X\backslash M)\cap M=\phi$ then there is in M a limit point of a sequence $\{p_k\}$ of points of $X\backslash M$ such that

1)′ *for each $k, \phi\neq L^{-}(p_k)$ and $p_{k+1}\notin L^{-}(p_k)$,* 2)′ *there exist a sequence* $\{p_{1n}\}$ *of points of* $X\backslash M$ *and a double sequence* $\{t_{kn}\}$ *of real numbers such that*

(i) *for each* $n, 0=t_{1n}>t_{2n}>\cdots>t_{kn}>t_{(k+1)n}>\cdots$,

(ii) *for each* $k\geqq 2$, $t_{kn}\rightarrow-\infty$ *as* $n\rightarrow\infty$,

(iii) *for each* $k, p_{1n}t_{kn}\rightarrow p_k$.

In the author's view, however, Bass' proof fails because of the necessity of making a countable infinity of choices of subsequences. Ura ([12], Theorem 2(10′), p. 284) has given a generalization of Bass' criterion to which, the author feels, in the absence of details of proof, the same criticism will apply. It should be observed that our Theorem 5 strengthens Bass' result in certain respects while weakening it in so far as it asserts the existence of certain convergent *subsequences* only.

The analysis now continues with a study of that part of $L^-(p)$, if any, outside M, where p is the point referred to in Theorem 1.

Let $Y\subseteq X$. We define three subsets of Y. Let $Q(Y)$ denote the union of non-dense, minimal saddle sets in Y. Define subsets, denoted $\mathcal{P}^{\pm}(Y)$, by

$\mathcal{P}^{\pm}(Y)=\{y\in Y: y\notin L^{\pm}(y),\ L^{\pm}(y)$ is a non-dense, saddle set$\}$.

Theorem 6. *Refer to Theorem* 1. *As regards the point* p, $L^-(p)\backslash M\subseteq Q(p)\cup\overline{\mathcal{P}^+(p)}$ *and* $L^-(p)\backslash M\subseteq Q(p)\cup\overline{\mathcal{P}^-(p)}$, *where* $Q(L^-(p)\backslash M)$ *has been abbreviated* $Q(p)$ *etc.*

For a proof we need the two following lemmas.

Lemma 4. *In a flow on a compact space if* $x\in L^+(x)$ *either* $L^+(x)$ *is minimal or there exist points* $y\in L^+(x)$ *such that* $\overline{\gamma^+(y)}\subset L^+(x)$. *Further the set of such points is dense in* $L^+(x)$.

Remark. The strict inclusion of $\overline{\gamma^+(y)}$ in $L^+(x)$ will be important.

A proof in the case of a continuous mapping of a compact space into itself, of which a discrete flow on a compact space is a special case, has been given by Friedlander ([14], Theorem 9) thus generalising and strengthening a result of Cherry ([3], Theorem VIII). This proof is valid, *mutatis mutandis*, in the case of a continuous flow.

Lemma 5. *If* $x, y\in X$ *and* $y\in L^+(x)$ *then* $y\in J^+(y)$ *and so* $y\in J^-(y)$.

The lemma states that points in limit sets are "non-wandering". For a proof see Ura ([12], Proposition 1, p. 287).

Proof of Theorem 6. If $L^-(p)\backslash M=\phi$ the result is trivial. In the contrary case let $r\in L^-(p)\backslash M$ so that $\phi\neq L^-(r)\subseteq L^-(p)$ and by Lemma 5, $r\in J^+(r)\cap J^-(r)$.

To begin with limit sets in $L^-(p)\backslash M$ are necessarily non-dense, since $L^-(p)$ is.

Concerning r there are three cases to consider.

(i) If $r\notin L^+(r)$ then $\phi\neq L^+(r)=L^+(r)\cap L^+(r)$ and $r\in J^+(r)\backslash L^+(r)$. Hence by Lemma 3, $L^+(r)$ is a saddle set. Thus $r\in\mathcal{P}^+(p)$.

(ii) If $r\in L^+(r)$ and $L^+(r)$ is minimal then $L^+(r)\cap M=\phi$; for otherwise $r\in M$. Since $m\notin L^+(r)$, $q\notin L^+(r)$ it follows from Proposition 4 (i) that $L^+(r)$ is a saddle set. Thus $r\in Q(p)$.

(iii) If $r\in L^+(r)$ and $L^+(r)$ is not minimal we use Lemma 4 applied to the subflow on the compact, invariant set $L^+(r)$. Thus there exists in any neighbourhood of r and so outside M, as we may suppose, a point s such that $\overline{\gamma^+(s)}\subset L^+(r)$. Hence $L^+(s)\subset L^+(r)$ and so $r\notin L^+(s)$. Further $\phi\neq L^+(s)=L^+(r)\cap L^+(s)$ and $r\in J^+(r)\backslash L^+(s)$. Hence by Lemma 3, $L^+(s)$ is a saddle set. Thus $s\in\mathcal{P}^+(p)$ and so $r\in\overline{\mathcal{P}^+(p)}$.

Finally it follows that $L^-(p)\backslash M\subseteq Q(p)\cup\overline{\mathcal{P}^+(p)}$. The same argument applied to negative limit sets and using sense variants of the lemmas shows that $L^-(p)\backslash M\subseteq Q(p)\cup\overline{\mathcal{P}^-(p)}$. Thus the theorem is proved.

The next theorem, a deduction from Theorem 6 relevant to a result of Ura, gives a necessary condition, stronger than Zubov's, for the positive instability of invariant sets with a certain "isolation" property.

Theorem 7. *Let M be positively unstable. Suppose there is a positive prolongation element E from M and a neighbourhood V of M such that non-dense, negative saddle limit sets in $E\cap V$ are in fact in M. Then for any neighbourhood U of M there exists a point $r\in(E\cap U)\backslash M$ such that $\phi\neq L^-(r)\subseteq M$.*

Proof. Let U be a neighbourhood of M. Then there exists a neighbourhood W of M such that $W\subseteq U\cap V$. By Theorem 1 there exists $p\in(E\cap W)\backslash M\subseteq(E\cap U)\backslash M$ such that $\phi\neq L^-(p)\subseteq E\cap W\subseteq E\cap V$. Further $L^-(p)$ is non-dense. There are two cases to consider.

(i) $L^-(p)\subseteq M$, in which case the theorem is proved ; take $r=p$;

(ii) $L^-(p)\backslash M\neq\phi$. Then $Q(p)=\phi$; for otherwise there exists a minimal set $K=L^-(K)\subseteq L^-(p)\backslash M\subseteq(E\cap V)\backslash M$, by Proposition 4 (i) a saddle set, contrary to hypothesis. Hence by Theorem 6, $L^-(p)\backslash M\subseteq\overline{\mathcal{P}^-(p)}$. Thus there exists a point $r\in L^-(p)\backslash M\subseteq(E\cap U)\backslash M$ such that $\phi\neq L^-(r)$ and $L^-(r)$ is a non-dense, saddle set in $L^-(p)$ and so in $E\cap V$. By hypothesis $L^-(r)\subseteq M$ and the theorem is proved.

This theorem generalizes for continuous flows one of Ura ([12], Theorem 3, p. 286), (or rather that part of it which is not an *a fortiori* consequence of the sufficiency of Zubov's condition for instability), viz, *if M is positively unstable and if there are, in some neighbourhood of M, no closed, invariant sets apart from those in M, then there exists a point $r\in X\backslash M$ such that $\phi\neq L^-(r)\subseteq M$.*

There is an important consequence of this theorem, whose easy proof we omit.

Theorem 8. *Let M be positively unstable so that positive prolongation elements from M exist. Then either there is a point outside M whose negative* *limit set is contained in M or for any positive prolongation element E of M and any neighbourhood V of M there is a non-dense, negative saddle limit set in $E\cap V$ not wholly contained in M.*

The first two remarks following Theorem 4 apply equally well here.

Theorem 9. *Let M be positively unstable suppose there is a positive prolongation element E from M and a neighbourhood V of M such that non-dense, positive saddle limit sets in $E\cap V$ are in fact in M. Then for any neighbourhood U of M there exists a point $r\in(E\cap U)\backslash M$ such that $\phi\neq L^-(r)\subseteq M$ or $\phi\neq L^+(r)\subseteq M$.*

Proof. Repeat the proof of Theorem 7 to the stage of showing that if $L^-(p)\backslash M\neq\phi$ then $Q(p)=\phi$. This merely requires the observation that $K=L^+(K)$. Then by Theorem 6, $L^-(p)\backslash M=\overline{\mathcal{P}^+(p)}$ and, using the same argument as in Theorem 7, there exists a point $r\in(E\cap U)\backslash M$ with $\phi\neq L^+(r)\subseteq M$. Hence the theorem is proved.

5. Planar continuous flows.

Standing hypotheses. Throughout this section we consider a continuous flow on a metrizable, orientable, (topological) 2-manifold X with the additional property that any Jordan curve C in X decomposes X into two connected, open sets D_1, D_2 which have C as common boundary i.e.

$$X\backslash C=D_1\cup D_2,\quad \partial D_1=C=\partial D_2.$$

We adopt Hájek's definition of a 2-manifold. See [6], p. 8 and the references cited there. By a Jordan curve is meant a homeomorph of the 1-sphere $\boldsymbol{S}^1$.

The Euclidean plane $\boldsymbol{R}^2$, the 2-sphere $\boldsymbol{S}^2$ and the cylinder $\boldsymbol{S}^1\times\boldsymbol{R}$ are amongst the 2-manifolds of the type considered, but not the 2-torus $\boldsymbol{S}^1\times\boldsymbol{S}^1$ nor the projective plane $\boldsymbol{P}^2$.

Proofs in the present section rely heavily on results contained in Hájek's recent book [6] concerning what he calls "continuous local dynamical systems on dichotomic carriers". The flows under consideration here form a special case of these and are described in Hájek's terms as "continuous global dynamical systems on metrizable, dichotomic carriers". The reader is referred to [6], in particular to Definitions 1.1 and 1.2 p. 44, 1.1 p. 92, 4.2 p. 173 and a parenthesized remark in 6.1 on p. 39. The metrizability of X is a non-redundant condition necessary for the application of earlier results of this paper.

Throughout this section $M\subseteq X$ shall be subject to the standing hypothesis of Section 4.

The following theorem clearly resolves in the affirmative, for flows on the class of 2-manifolds under present consideration, the conjecture proposed in

Section 1.

Theorem 10. *Let M be positively unstable. Suppose there is a positive prolongation element E from M and a neighbourhood V of M such that $(E\cap V)\backslash M$ contains no saddle fixed points. Then for any neighbourhood U of M there exists a point $r\in(E\cap U)\backslash M$ such that $\phi\neq L^{-}(r)\subseteq M$.*

Before giving a proof of this theorem we need two lemmas.

Lemma 6. *A cycle is not a saddle set.*

Proof. Suppose to the contrary that C were a cycle and a saddle set. Let D_1, D_2 denote the two components of $X\backslash C$. Consider the flows obtained by restriction (in the obvious sense) of the flow on X to $\bar{D}_1$ and $\bar{D}_2$. (Note that $\bar{D}_1$ and $\bar{D}_2$ are both invariant.) Then C, being a saddle set, is a saddle set and so unstable in both senses as regards one at least of these flows, say that on $\bar{D}_1$. On the other hand C is stable in at least one sense as regards the flow on $\bar{D}_1$. This follows from what Hájek calls the "cycle stability theorem" ([6], 3.3, p. 196), which asserts basically that a cycle C is stable in at least one sense as regards the flows on both $\bar{D}_1$ and $\bar{D}_2$ and elaborates on the nature of its stability. In view of this contradiction the result follows.

Concerning the cycle stability theorem the reader should check in particular that orbital stability in the sense of Hájek is equivalent to stability in the sense of this paper. See Definition 4.9 and Remarks 4.10 on p. 113 of [6].

For the same result in a different context see Ura ([12], Lemma 1, p. 291).

Lemma 7. *If $x\in X$ then $L^{+}(x)$ is a cycle or, for all $y\in L^{+}(x)$, $L^{+}(y)\cup L^{-}(y)$ consists entirely of fixed points.*

This is, in our language and notation, Proposition 1.11 on p. 184 of [6].

Proof of Theorem 10. Let U be a neighbourhood of M. Then there exists a neighbourhood W of M such that $W\subseteq U\cap V$. By Theorem 1 there exists $p\in(E\cap W)\backslash M\subseteq(E\cap U)\backslash M$ such that $\phi\neq L^{-}(p)\subseteq E\cap W\subseteq E\cap V$. There are two cases to consider.

(i) $L^{-}(p)\subseteq M$, in which case the theorem is proved; take $r=p$.

(ii) $L^{-}(p)\backslash M\neq\phi$, in which case $L^{-}(p)$ is not a cycle. For if so, since $L^{-}(p)$ is minimal, $L^{-}(p)\cap M=\phi$ and, by Proposition 4 (i), $L^{-}(p)$ is a saddle set, contradicting Lemma 6. It follows from Lemma 7 that, for any $r\in L^{-}(p)\backslash M$, $L^{-}(r)\neq\phi$ consists entirely of fixed points. Should any of these points lie outside M they are, by Proposition 4(i), saddle fixed points in $L^{-}(r)\backslash M\subseteq(E\cap V)\backslash M$; contradicting the hypothesis. Thus $\phi\neq L^{-}(r)\subseteq M$ for any $r\in L^{-}(p)\backslash M$ and the theorem is proved.

Remark. When the second case obtains, $\phi\neq L^{+}(r)\subseteq M$ for all $r\in L^{-}(p)\backslash M$, as a similar proof shows.

This theorem generalizes and strengthens for continuous flows a result of

Ura ([12], Corollary, p. 292), (or rather that part of it which is not an *a fortiori* consequence of the sufficiency of Zubov's condition for instability), viz, *for a continuous flow on* $\boldsymbol{R}^2$, *if* M *is positively unstable and if in some neighbourhood of* M *there are no saddle fixed points then* $L^-(X\backslash M)\cap M\neq\phi$.

6. Existence of Lagrange stable semi-trajectories.

In this section we point out that a Lagrange stable semi-trajectory exists outside of but in any neighbourhood of a non-open, closed, invariant set having a neighbourhood with compact closure. To the author's knowledge the result in this generality does not appear explicitly in the literature although it is known when M is a singular point of a planar dynamical system, as noted below.

Theorem 11. *Let* M *be a non-open, closed, invariant set, having a neighbourhood with compact closure. In any neighbourhood of* M *there exists a Lagrange stable semi-trajectory outside* M.

The proof follows from Theorem 1 when M is positively unstable and, when M is positively stable, from the fact that there are points of $X\backslash M$ arbitrarily close to M, since M is not open.

Alternatively, for continuous flows at least, the theorem is an immediate consequence of a result of Ura and Kimura ([13], Théorème 3).

There is an interesting corollary which we now consider.

It is well known that a point of X is fixed if there is a semi-trajectory in any punctured neighbourhood of it. See, for example, Nemytskii and Stepanov ([8], 2.10, p. 332). As far as the author is aware, the converse, now to be deduced, is known only for planar dynamical systems. See, for example, Sansone and Conti ([10], Theorem 26, p. 178). Indeed Mendelson [7] has given a sufficient condition for a fixed point of a dynamical system to have a semi-trajectory in any punctured neighbourhood of it. This condition is now seen to be superfluous.

Corollary. *There is a semi-trajectory in any punctured neighbourhood of a fixed point* x *which is an accumulation point of* X *and has a neighbourhood with compact closure.*

For a proof apply the theorem to $\{x\}$.

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