Funkcialaj Ekvacioj, **13** (1970), 51–59

On Algebraic Differential Equations

By Yasutaka SIBUYA[1)]

(University of Minnesota)

Dedicated to Professor Tokui Satō on the Occasion of His Retirement

1. Introduction.

We shall consider a differential equation

$$\text{(E)} \qquad F(x, y, y') \equiv P(x, y)(y')^2 + 2Q(x, y)y' + R(x, y) = 0,$$

where $y' = dy/dx$, and P, Q and R are polynomials in y whose coefficients are holomorphic in x in a neighborhood of $x=0$. Assume that a solution $y=y(x)$ admits an essential singularity ω at $x=0$. Then, according to a theorem of T. Kimura [1], the solution $y(x)$ takes all complex values (other than a finite number of possible exceptional values) in every small neighborhood of ω. The exceptional values are determined by P, Q and R explicitly. Let y_0 be different from such an exceptional value. Then by virtue of Kimura's theorem, there exists a sequence $\{x_n\}$ such that $x_n \to \omega$ as $n \to \infty$ and $y(x_n) = y_0$ for every n. However, this does not mean that $y'(x)$ takes all possible values. In other words, even if $p = \varphi(x)$ is a root of $F(x, y_0, p) = 0$, there may not exist any sequence $\{x_n\}$ such that $x_n \to \omega$ as $n \to \infty$, $y(x_n) = y_0$ and $y'(x_n) = \varphi(x_n)$ for every n.

To illustrate such a situation, we shall consider the differential equation :

$$(1.1) \qquad x(y')^2 + 2yy' + y^3 = 0.$$

If y is bounded, two roots of $xp^2 + 2yp + y^3 = 0$ are

$$p = \varphi_1(x, y) = -x^{-1}y\left\{2 + \sum_{n=1}^{\infty} \alpha_n (xy)^n\right\}$$

and

$$p = \varphi_2(x, y) = y^2 \sum_{n=1}^{\infty} \alpha_n (xy)^{n-1},$$

where

$$\left(1 + \sum_{n=1}^{\infty} \alpha_n (xy)^n\right)^2 = 1 - xy.$$

If, for y_0, there exists a sequence $\{x_n\}$ such that

1) This paper was written with partial support from the National Science Foundation under Grant No. GP-7041 X. The author wishes to express his appreciation to Professor Tosiya Saito for his encouragement and advice during the preparation of this paper.

(1.2)
$$\begin{cases} x_n \to \omega & \text{as} \quad n \to \infty, \\ y(x_n) = y_0 & \text{for every } n, \\ y'(x_n) = \varphi_2(x_n, y_0) & \text{for every } n, \end{cases}$$

the solution $y(x)$ must be holomorphic at $x=0$. On the other hand, by solving $y'=\varphi_1(x, y)$, we can find a solution of (1.1) which admits an essential singularity ω at $x=0$. For this solution, the situation (1.2) is impossible in the neighborhood of ω.

Let us construct another example. It can be shown that a differential equation of the form

$$xz' = x\left\{1+\sum_{n=1}^{\infty} a_n z^{2n}\right\} + z^3,$$

a_n being constants, has a solution $z=z(x)$ which admits an ordinary transcendental singularity ω at $x=0$, and that $z(x)\to 0$ as $x\to\omega$. Furthermore, $y(x)=x^{-1}z(x)$ admits an essential singularity at ω. Keeping this remark in mind, consider the equation

(1.3)
$$(xy'+y-x^2y^3)^2 = 1+(xy)^2.$$

Putting $z=xy$, we derive from (1.3) the equation

$$(z'-x^{-1}z^3)^2 = 1+z^2.$$

Let

$$1+z^2 = \left(1+\sum_{n=1}^{\infty} \alpha_n z^{2n}\right)^2,$$

where α_n are constants, and consider

$$xz' = x\left\{1+\sum_{n=1}^{\infty} \alpha_n z^{2n}\right\} + z^3.$$

As we mentioned above, this equation has a solution $z=z(x)$ which admits an ordinary transcendental singularity ω at $x=0$, and $z(x)\to 0$ as $x\to\omega$. Furthermore, $y(x)=x^{-1}z(x)$ is a solution of (1.3) which admits an essential singularity at ω. Note that

$$xy'+y-x^2y^3 = 1+\sum_{n=1}^{\infty} \alpha_n (xy)^{2n},$$

in the neighborhood of ω. This means that, in the neighborhood of ω, $y(x)$ does not satisfy

$$xy'+y-x^2y^3 = -1-\sum_{n=1}^{\infty} \alpha_n (xy)^{2n}$$

which is another branch of (1.3).

It must be clearly remarked that *the solution* $y(x)$ *of* (1.3) *which was constructed above admits an essential singularity* ω *at* $x=0$, *but* $z(x)=xy(x)$ *admits an ordinary transcendental singularity at* ω. Let us call such a singularity a singularity of calss (A). A precise definition of singularities of class (A) will be given in Section 2. The purpose of the present work is to show that, if ω is *not of class* (A), then not only $y(x)$ but also $y'(x)$ take all possible values in every neighborhood of ω. In other words (and very roughly speaking), we claim that, if a singularity ω is not of class (A), then the point $(y(x), y'(x))$ moves almost all over the Riemann surface $F(x, y, p)=0$ in every small neighborhood of ω.

2. Main theorem.

A rational function $H(x,y,p)$ in y and p is said to be non-constant on the Riemann surface $F(x,y,p)=0$ for each fixed x, if there is no function $a(x)$ of x such that $H(x,y,p)\equiv a(x)$ for $F(x,y,p)=0$. Assume that a solution $y(x)$ of (E) admits a singularity ω at $x=0$. *The singularity* ω *is said to be of class* (A), *if there exists a rational function* $H(x,y,p)$ *in* y *and* p, *which is non-constant on the Riemann surface* $F(x,y,p)=0$ *for each fixed* x *and whose coefficients are holomorphic at* $x=0$, *such that* $H(x,y(x), y'(x))$ *admits at most an ordinary transcendental singularity at* ω. For example, the solution $y(x)$ of (1.3) which was constructed in Section 1 admits an essential singularity ω of class (A) at $x=0$. To see this, it is sufficient to put $H(x, y, p)=xy$. In general, if $F(x, y,p)$ is irreducible with respect to y and p, and if $y(x)$ admits an essential singularity ω at $x=0$ and if $y(x)$ admits only a finite number of branches around ω, then ω is not of class (A). In fact, if it were of class (A), there would be a rational function $H(x,y,p)$ in y and p such that

(i) its coefficients are holomorphic at $x=0$,

(ii) it is non-constant on the Riemann surface $F(x,y,p)=0$ for each fixed x,

(iii) $H(x,y(x),y'(x))$ admits at most an ordinary transcendental singularity at ω.

Since $y(x)$ admits only a finite number of branches at ω, $H(x, y(x), y'(x))$ can not admit any transcendental singularity at ω. Put $\lambda(x)=H(x, y(x), y'(x))$. Then

$$\lambda(x) = H(x, y, p)$$

is not an identity on the surface $F(x,y,p)=0$. Let us eliminate p from $\lambda(x)=H(x,y,p)$ and $F(x,y,p)=0$ to obtain a non-trivial relation $G(x,y)=0$. This is possible, since F is irreducible. It is clear that G is a polynomial in y whose coefficients can not admit any transcendental singularity at ω. However, this is impossible, since $y(x)$ admits an essential singularity at ω. This proves that ω is not of class (A). The reasoning given above was used by J. Malmquist [3] in his study of algebraic differential equations.

Equation (E) can be rewritten as

$$(P(x, y)y'+Q(x, y))^2-D(x, y)=0, \tag{E}$$

where

$$D(x, y)=Q(x, y)^2-P(x, y)R(x, y).$$

If we assume that P and Q may admit poles with respect to x at $x=0$, we can assume without loss of generality that $D(0, y)\not\equiv 0$. Assume that $D(0, y_0)\neq 0$ and let

$$q=\varphi(x, y_0)=\alpha(y_0)+O(x) \quad \text{and} \quad q=-\varphi(x, y_0)$$

be two roots of $q^2=D(x, y_0)$, where $(\alpha(y_0))^2=D(0, y_0)$.

Now we can state our main theorem.

Theorem. *Assume that a solution $y(x)$ of* (E) *admits an essential singularity ω at $x=0$ and that ω is not of class* (A). *Then there exist two sequences $\{x_{n,1}\}$ and $\{x_{n,2}\}$ such that*

$$\begin{cases} x_{n,j}\to\omega \text{ as } n\to\infty, \quad j=1, 2, \quad y(x_{n,j})=y_0, \quad j=1, 2, \\ P(x_{n,1}, y_0)y'(x_{n,1})+Q(x_{n,1}, y_0)=\varphi(x_{n,1}, y_0), \\ P(x_{n,2}, y_0)y'(x_{n,2})+Q(x_{n,2}, y_0)=-\varphi(x_{n,2}, y_0), \\ \qquad\qquad n=1, 2, \cdots, \end{cases} \tag{2.1}$$

if y_0 is different from a finite number of exceptional values.

3. An example.

It was shown in Section 1 that, if a solution $y(x)$ of (1.1) admits a singularity ω at $x=0$, then such a situation as (1.2) is impossible in the neighborhood of ω. If the assertion of our theorem is true, then every singularity ω at $x=0$ of a solution $y(x)$ of (1.1) must be of class (A). In this section, we shall prove that this is actually the case. The proof of our main theorem which will be given in Sections 4 and 5 will be very similar to the proof given in this section.

Equation (1.1) can be rewritten as

$$(xy'+y)^2-(y^2-xy^3)=0. \tag{3.1}$$

Let us put

$$w=(y-y_0)^{-1}(q+y_0\varphi(x, y_0)),$$

where (x, y, q) satisfies

(3.2) $$q^2=y^2-xy^3$$

and

$$\varphi(x, y_0)=1+\sum_{n=1}^{\infty}\alpha_n(xy_0)^n, \quad (\varphi(x, y_0))^2=1-xy_0,$$

and α_n are constants. Note that

$$\begin{aligned} w^2&=(y-y_0)^{-2}\{q^2+2y_0q\varphi(x, y_0)+y_0^2(\varphi(x, y_0))^2\} \\ &=(y-y_0)^{-2}\{y^2-xy^3+2y_0q\varphi(x, y_0)+y_0^2(\varphi(x, y_0))^2\}, \end{aligned}$$

and that $(y-y_0)^{-2}q$ is bounded as $|y|\to\infty$. Therefore, if we put

(3.3) $$v=w^2+xy,$$

we can prove that v is bounded if (x, y, q) is on the surface (3.2) and $|y|$ is sufficiently large. Now assume that v tends to infinity under the assumption that (x, y, q) is on the surface (3.2). This means that w tends to infinity, but y remains bounded. Hence y tends to y_0. Thus we get

$$q=y\varphi(x, y) \quad \text{or} \quad q=-y\varphi(x, y).$$

If $q=-y\varphi(x, y)$, then $q+y_0\varphi(x, y_0)=O(|y-y_0|)$, and hence w must be bounded. Therefore, if v tends to infinity, we must have $q=y\varphi(x, y)$ and $y\to y_0$.

Now we claim that

$$v(x)=H(x, y(x), y'(x)),$$

where

$$H(x, y, p)=(y-y_0)^{-2}\{xp+y+y_0\varphi(x, y_0)\}^2+xy,$$

is bounded in the neighborhood of ω. Note that

$$(xp+y)^2=y^2-xy^3$$

if $y=y(x)$ and $p=y'(x)$. If $v(x)$ were not bounded in the neighborhood of ω, there would be a sequence $\{x_n\}$ such that $x_n\to\omega$ as $n\to\infty$, and $v(x_n)\to\infty$. Then $y(x_n)\to y_0$ and $x_ny'(x_n)+y(x_n)=y(x_n)\varphi(x_n, y(x_n))$, or

$$\begin{gathered} x_n\to\omega \text{ as } n\to\infty, \\ y(x_n)\to y_0 \text{ as } n\to\infty, \\ y'(x_n)=(y(x_n))^2\sum_{m=1}^{\infty}\alpha_m(x_ny(x_n))^{m-1} \\ \text{for } n=1, 2, \cdots. \end{gathered}$$

Then $y(x)$ must be holomorphic at ω. This is a contradiction. Therefore $v(x)$ is bounded in the neighborhood of ω. On the other hand, it is easily shown that $v(x)$ satisfies an algebraic differential equation. Hence by virtue of Kimura's theorem [1], $v(x)$ can not admit an essential singularity at ω. This proves that ω is of class (A).

The construction of $H(x, y, p)$ amounts to a construction of an analytic function on the Riemann surface $F(x, y, p)=0$ which admits a pole only at a given point. A difficulty arises from the fact that the Riemann surface depends on an extra parameter x. We must study the behavior of such an analytic function as $x\to 0$. The construction given above was derived from the addition formula for Weierstrass elliptic function $\wp(u)$:

$$\wp(u+a)=\frac{1}{4}\left(\frac{\wp'(u)-\wp'(a)}{\wp(u)-\wp(a)}\right)^2-\wp(u)-\wp(a).$$

Roughly speaking, by replacing $\wp(u+a)$, $\wp(u)$, $\wp(a)$, $\wp'(u)$ and $\wp'(a)$ by v, y, y_0, q and q_0 respectively, we arrive at the definition of $v(x)$ given above.

An application of such an analytic function as $H(x, y, p)$ to the study of algebraic differential equations was made by J. Malmquist [3]. An expository treatment of the global theory of algebraic differential equations has been given by T. Kimura [2].

4. Proof of main theorem : Part I.

We shall prove the existence of $\{x_{n,1}\}$. To do this, consider a surface defined by

$$q^2=D(x, y). \tag{4.1}$$

Define $\varphi(x, y_0)$ in the same way as in Section 2, and put

$$u=\varphi(x, y_0)+q, \tag{4.2}$$

where (x, y, q) is on the surface (4.1). Let

$$u^m=R_m(x, y)+S_m(x, y)q \qquad (m=1, 2, \cdots), \tag{4.3}$$

where R_m and S_m are polynomials in y whose coefficients are holomorphic at $x=0$. We shall prove that

$$S_m(0, y_0)\neq 0 \qquad \text{for } m=1, 2, \cdots. \tag{4.4}$$

To do this, note that at $u^{m+1}=u^m u$ implies

$$R_{m+1}(x, y)=R_m(x, y)\varphi(x, y_0)+S_m(x, y)D(x, y),$$
$$S_{m+1}(x, y)=R_m(x, y)+S_m(x, y)\varphi(x, y_0).$$

Hence

$$\begin{aligned}R_{m+1}(0, y_0)&=R_m(0, y_0)\varphi(0, y_0)+S_m(0, y_0)D(0, y_0)\\&=R_m(0, y_0)\varphi(0, y_0)+S_m(0, y_0)(\varphi(0, y_0))^2\\&=\varphi(0, y_0)\{R_m(0, y_0)+S_m(0, y_0)\varphi(0, y_0)\}\end{aligned}$$

$$=\varphi(0, y_0)S_{m+1}(0, y_0).$$

Thus we obtain

$$S_{m+1}(0, y_0)=2S_m(0, y_0)\varphi(0, y_0) \qquad (m=1, 2, \cdots).$$

Since $S_1(x, y)\equiv 1$, we can prove (4.4) by induction.

Now let us put

$$w=\frac{u}{y-y_0} \tag{4.5}$$

to obtain

$$w^m=\frac{R_m(x, y)}{(y-y_0)^m}+q\frac{S_m(x, y)}{(y-y_0)^m}. \tag{4.6}$$

The coefficient of q can be written as

$$\frac{S_m(x, y)}{(y-y_0)^m}=A_m(x, y)+\sum_{k=1}^{m}\frac{a_{m,k}(x)}{(y-y_0)^k}, \tag{4.7}$$

where $A_m(x, y)$ is a polynomial in y whose coefficients are holomorphic in x at $x=0$, and $a_{m,k}(x)$ are holomorphic at $x=0$. It is easily seen that we have

$$a_{m,m}(0)=S_m(0, y_0)\neq 0 \qquad (m=1, 2, \cdots). \tag{4.8}$$

Denote by d the degree of $D(x, y)$ with respect to y, and put

$$g=\begin{cases}\frac{1}{2}d+1 & \text{if } d \text{ is even,}\\ \frac{1}{2}(d+1) & \text{if } d \text{ is odd.}\end{cases} \tag{4.9}$$

Then $q(y-y_0)^{-g}$ is bounded as y tends to infinity.

Observe that

$$w^g=(y-y_0)^{-g}R_g(x, y)+A_g(x, y)q+\sum_{k=1}^{g}a_{g,k}(x)(y-y_0)^{-k}q$$

and

$$w^{g-1}=(y-y_0)^{-g+1}R_{g-1}(x, y)+A_{g-1}(x, y)q+\sum_{k=1}^{g-1}a_{g-1,k}(x)(y-y_0)^{-k}q.$$

Hence

$$\begin{aligned}w^g-\left(\frac{a_{g,g-1}(x)}{a_{g-1,g-1}(x)}\right)w^{g-1}&=B_g(x, y)+C_g(x, y)q\\&+\sum_{k=1}^{g-2}b_k(x)(y-y_0)^{-k}q\\&+c(x)q(y-y_0)^{-g},\end{aligned}$$

where B_g is a rational function of y with holomorphic coefficients, C_g is a polynomial in y with holomorphic coefficients, and b_k and c are holomorphic at x

$=0$. Note that the coefficient of w^{g-1} on the left side is holomorphic at $x=0$ by virtue of (4.8).

In this manner, we can find functions $\mu_1(x), \cdots, \mu_{g-1}(x)$ such that they are holomorphic in x at $x=0$ and that

$$w^g+\sum_{k=1}^{g-1}\mu_k(x)w^k=A(x,y)+B(x,y)q+c(x)q(y-y_0)^{-g},$$

where A is a rational function of y with holomorphic coefficients, and $B(x,y)$ is a polynomial in y with holomorphic coefficients. Let $A(x, y)=A_0(x, y)+O(|y|^{-1})$, where A_0 is a polynomial in y with holomorphic coefficients. Then define a rational function $K(x,y,q)$ by

$$K(x,y,q)=w^g+\sum_{k=1}^{g-1}\mu_k(x)w^k-A_0(x,y)-B(x,y)q. \tag{4.10}$$

Then, K is bounded if $|y|$ is sufficiently large. Therefore, if $K\to\infty$, then y is bounded. Hence w must tend to infinity. This implies that $y\to y_0$. If $|x|$ is sufficiently small, either $q=\varphi(x, y)$ or $q=-\varphi(x, y)$. If $q=-\varphi(x, y)$, then $u=\varphi(x, y_0)+q=O(|y-y_0|)$, and hence K is bounded. Therefore, if $K\to\infty$, we must have

$$y\to y_0 \text{ and } q=\varphi(x,y).$$

Now define a rational function $H(x,y,p)$ by

$$H(x,y,p)=K(x,y,P(x,y)p+Q(x,y)), \tag{4.11}$$

and put

$$v(x)=H(x,y(x),y'(x)). \tag{4.12}$$

5. Proof of main theorem : Part II.

We shall prove now that $v(x)$ satisfies an algebraic differential epuation. Let

$$q(x)=P(x,y(x))y'(x)+Q(x,y(x)). \tag{5.1}$$

Then

$$q'(x)=\frac{1}{2}q(x)^{-1}D_x(x,y(x))+D_y(x,y(x))y'(x), \tag{5.2}$$

where $D_x=\partial D/\partial x$ and $D_y=\partial D/\partial y$. From (5.1) and (5.2) we derive

$$q'(x)=r(x,y(x))+s(x,y(x))q(x), \tag{5.3}$$

where $r(x,y)$ and $s(x,y)$ are rational in y with holomorphic coefficients. Let us write $v(x)$ in the form :

$$v(x)=V(x,y(x))+U(x,y(x))q(x), \tag{5.4}$$

where V and U are rational in y with holomorphic coefficients. Then we get

$$v'(x)=W(x,y(x))+Z(x,y(x))q(x), \tag{5.5}$$

where W and Z are rational in y with holomorphic coefficients. Since $(q(x))^2 = D(x,y(x))$, by eliminating $q(x)$, we obtain two relations

$$F_1(x,v(x),y(x))=0 \tag{5.6}$$

and

$$F_2(x,v'(x),y(x))=0, \tag{5.7}$$

where F_1 and F_2 are polynomials in (v, y) and in (v', y) respectively and their coefficients are holomorphic at $x=0$. Both of them are either quadratic or linear in v and v' respectively. Hence by eliminating $y(x)$, we obtain an algebraic differential equation for $v(x)$.

By virtue of Kimura's theorem [1], $v(x)$ can not admit an essential singularity at ω if $v(x)$ is bounded. If $v(\xi_n)$ tend to infinity, where $\{\xi_n\}$ is a sequence such that $\xi_n\to\omega$ as $n\to\infty$, then $y(\xi_n)\to y_0$ as $n\to\infty$, and $q(\xi_n)=\varphi(\xi_n, y(\xi_n))$ for large n. Hence if y_0 is not exceptional in the sense of Kimura, we can find a sequence $\{x_n\}$ such that

$$x_n \to \omega \quad \text{as} \quad n\to\infty,$$
$$y(x_n)=y_0 \quad \text{for every } n,$$
$$q(x_n)=\varphi(x_n, y_0) \quad \text{for every } n,$$

in the same manner as in Kimura's paper [1]. This completes the proof of the main theorem.

References

[1] T. Kimura, Sur les points singuliers des équations différentielles ordinaires du premier ordre, Comment. Math. Univ. St. Paul, **2** (1954) 47–53.

[2] T. Kimura, On the global theory of algebraic differential equations, Proc. U.S.-Japan Seminar on Differential and Functional Equations, pp 181–197, Benjamin, 1967.

[3] J. Malmquist, Sur les fonctions à un nombre fini de branches satisfaisant à une équation différentielle du premier ordre, Acta Math. **74** (1941), 175–196.

(Ricevita la 12-an de decembro, 1969)