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A Continuous Differential Equation in Hilbert Space without Existence

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Let B be a Banach space and let $F: R\times B\rightarrow B$ be continuous. It is known that if either B is finite dimensional or if F satisfies a Lipschitz condition, then for each $(t_0, x_0)\in R\times B$ there exists a C^1 function $x(t)$ with values in B defined for t in an interval neighborhood of $I\subset R$ of t_0 such that $x(t_0)=x_0$ and

$$\dot{x}(t)=F(t, x(t)), \quad (\text{“}\cdot\text{” means “}\frac{d}{dt}\text{”}). \tag{1}$$

Dieudonné [1], [2], gave a very simple example where $B=(c_0)$, (the space of real-valued sequences $x=(x_1, x_2, \cdots)$ with $x_n\rightarrow 0$ as $n\rightarrow\infty$, where $\|x\|_{c_0}=\sup_n|x_n|$). He points out that for $x=(x_1, x_2, \cdots)$, if the n^{th} coordinate of $F(x)$ is $|x_n|^{1/2}+n^{-1}$, then F is continuous in (c_0) and there exists no solution $x(t)$ in (c_0) such that $x(0)=(0)$. Actually his equation has *no solutions at all* in (c_0) ; the n^{th} coordinate x_n satisfies the one dimensional equation $\dot{x}_n=|x_n|^{1/2}+n^{-1}$, the solution of which is increasing more rapidly than the maximal solution of $\dot{r}=|r|^{1/2}$, with initial condition $r(t_0)=x_n(t_0)$, ([3], p. 25, see proof of Lemma 2.1). If for some t_0, $\lim_{n\rightarrow\infty} x_n(t_0)=0$, then

$$\lim_{n\rightarrow\infty} x_n(t_0+\tau)=(\text{sign } \tau)\tau^2/4, \quad t_0+\tau\in\text{domain } x\, ; \tag{2}$$

that is, if $x(t_0)\in(c_0)$, then $x(t+\tau)\notin(c_0)$, (unless $\tau=0$). This example seems to depend strongly on the properties of (c_0) (which is not reflexive). See Remark (ii). As far as I know, no example has been published even where B is a Hilbert space. I now give such an example with no solution such that $x(0)=0$.

Let H be the Hilbert space of sequences of real numbers, $y=(y_1, y_2, \cdots)$, such that $\|y\|^2=\sum y_i^2$. Let P_n be the projections given by $P_n(y)=(0, \cdots, 0, y_{n+1}, y_{n+2}, \cdots)$, $n=1, 2, \cdots$ and $P_0(y)=y$. For $t\in R$, $y\in H$, define

$$P(t)y=\begin{cases} 0 & \text{for } t\leqq 0,\\ y & \text{for } t\geqq 1,\\ (2-2^n t)P_n y+(2^n t-1)P_{n-1}y & \text{for } t\in[2^{-n}, 2^{-n+1}],\ n=1, 2, \cdots. \end{cases}$$

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Claim. P is continuous on $R\times H$. P is clearly continuous at (t, x) if $t\neq 0$. For continuity at $(0, x)$, let $t_i\to 0$ and let y_i be a sequence of points in H with $y_i\to x$. Write $y_i=(y_{i1}, y_{i2}, \cdots)$. Then for each n

$$\|P(t_i)y_i\|^2\leqq \sum_{j=n}^{\infty} y_{ij}^2 \qquad \text{for } t_i\leqq 2^{-n+1}, \quad \text{and} \quad \limsup_{i\to\infty}\|P(t_i)y_i\|\leqq \sum_{j=n}^{\infty} x_j^2.$$

Since $\sum_{j=n}^{\infty} x_i^2$ can be made as close to 0 as desired by choosing n large, $\limsup_{i\to\infty} \|P(t_i)y_i\|=0$, proving the claim Let

$$G(y)=y\|y\|^{-1/2} \text{ for } y\neq 0 \quad \text{and} \quad G(0)=0.$$

Define $A(y)=(|y_1|,|y_2|, \cdots)$, and let $v=(2^{-1}, 2^{-2}, 2^{-3}, \cdots)$ and

$$F(t, y)=G(P(t)A(y))+P(t/2)v \max\{0, 4^{-1}t^2-\|y\|\}.$$

Note that $F\colon R\times H\to H$ is continuous. Write $F=(F_1, F_2, \cdots)$.

Claim. There is no solution $x(\cdot)$ of (1), whose domain I is an open interval containing 0, such that $x(0)=0$. We now suppose the contrary; assume that $x(\cdot)$ is such a solution. Write $x(t)=(x_1(t), \cdots)$. Since $P(t)=0$ for $t\leqq 0$, we have $x(t)=0$ for $t\leqq 0$. From the definition of $A(\cdot)$, $F_n(t, y)\geqq 0$ for all t, y and n, so each $x_n(t)$ is non-decreasing and $x_n(t)\geqq 0$ for $t\in I$, so $A(x(t))=x(t)$; also $F(t,0)=4^{-1}t^2P(t/2)v\neq 0$ for $t>0$, so $x(t)\neq 0$ for $t>0$. For $n=1, 2, \cdots$ and $t\leqq 2^{-n}$, $F_n(t, y)\equiv 0$ and $\dot{x}_n(t)\equiv 0$, so $x_n(t)=0$ for $t\leqq 2^{-n}$.

Now write $\gamma(t)=\|x(t)\|^{1/2}$. Then for our solution $x(\cdot)$, γ is continuous and for each n,

$$\dot{x}_n(t)=\frac{(2^n t-1)x_n(t)}{\gamma(t)} \quad \text{for } t\in[2^{-n}, 2^{-n+1}];\ x_n(2^{-n})=0.$$

Hence on $[2^{-n}, 2^{-n+1}]$ the coordinate x_n must be 0; therefore, for all t, $P(t)\cdot A(x(t))=x(t)$. Let $\rho(t)=\|x(t)\|^2$. Then letting $\langle x, y\rangle=\sum x_i y_i$,

$$\text{(3)} \qquad \frac{d}{dt}\rho(t)=2\langle x(t),\dot{x}(t)\rangle=2\|x(t)\|^{3/2}+2\langle x(t), v\rangle\max\{\ ,\ \}$$
$$\geqq 2\|x(t)\|^{3/2}=2\rho^{3/4}(t) \qquad t>0,\ t\in I.$$

Then $\rho(t))\geqq(t/2)^4$ for $t>0$, $\|x(t)\|\geqq t^2/4$ for $t\geqq 0$, and $\max\{0, t^2/4-\|x(t)\|\}=0$. Hence our particular solution $x(t)$ satisfies (for $t\in I$)

$$\dot{x}(t)=x(t)\|x(t)\|^{-1/2},\ t>0;\ \text{that is,}$$

$$\text{(4)} \qquad \dot{x}_n(t)=x_n(t)\rho(t)^{-1/4},\ x_n(2^{-n})=0, \text{ for } n=1, 2, \cdots;\ t>0.$$

The solution of (4) for each n (for any continuous scalar function $\rho(t)>0$) is $x_n(t)\equiv 0$; hence $x(t)\equiv 0$, contradicting our earlier result. Therefore no solution $x(t)$ can exist with $x(0)=0$.

Remarks. (i) This example can be extended to many Banach spaces, including Banach spaces of sequences of real numbers, such as ℓ_p, $1 \leqq p \leqq \infty$, in which "$\|(x_1, x_2, \cdots)\| \leqq \|(y_1, y_2, \cdots)\|$" is implied by "$|x_n| \leqq |y_n|$ for all n". This fact (not the existence of an inner product) is needed to guarantee (3) $\dot{\rho} \geqq 2\rho^{3/4}$ for almost all $t \in I$. Note that ρ is absolutely continuous. For $x \in \ell_p$, F is continuous without any changes and there is no solution with $x(0)=0$.

(ii) Although (c_0) is complete, its unit ball is not compact in the weak topology induced by its dual, which is ℓ_1. The dual of ℓ_1 is ℓ_∞ (which contains (c_0)). Dieudonné's equation is also defined on ℓ_∞, and since there exists no continuous projection of ℓ_∞ onto (c_0), there is no obvious way to restrict his equation to only (c_0). Curiously, Dieudonné's equation has a solution in ℓ_∞ (for any initial $x(0)$ in (c_0) or even in ℓ_∞), and for each $x \in (c_0)$ and each t, $x(t)$ is the weak limit of a sequence in (c_0). Since H is reflexive, its unit ball is weakly compact, so no analogous problems arise in the example presented here.

Bibliography

[1] J. Dieudonné, Foundation of Modern Analysis, Academic Press, New York, 1960, see p. 287, in first edition.

[2] J. Dieudonné, Deux exemples singuliers d'équations différentielles, Acta Scien. Math. (Szeged) **12** (1950), B 38–40.

[3] P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964.

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