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A Relationship between Uniformly Asymptotic Stability and Total Stability

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Dedicated to Professor Tokui Satō on the Occasion of His Retirement

It is well known that for linear systems total stability is equivalent to uniformly asymptotic stability and that for nonlinear systems, if the Liapunov function which characterizes the uniformly asymptotic stability satisfies a uniform Lipschitz condition, uniformly asymptotic stability implies total stability (cf. [2]). Miller [1] has shown that if an almost periodic system of ordinary differential equations has a bounded solution which is totally stable, then the system has an almost periodic solution, and recently, one of the authors [3] has shown the existence of an almost periodic solution for functional-differential equations under the assumption that a bounded solution is uniformly asymptotically stable. Both of them have assumed that the solution is unique for any system in the hull. In this article, by using a technique similar to the one in [3], we shall show that for more general systems, the uniformly asymptotic stability of a bounded solution implies total stability.

Consider a system of functional-differential equations

$$\dot{x}(t)=F(t, x_t), \tag{1}$$

and assume that $F(t,\varphi)$: $I\times\bar{C}_{B^*}\to R^n$ is continuous and $|F(t,\varphi)|\leqq L$ for a constant $L>0$ and all $(t,\varphi)\in I\times\bar{C}_{B^*}$, where R^n is the Euclidean n-space, $I=[0,\infty)$ and $\bar{C}_{B^*}$ denotes the set of all continuous vector functions φ defined on $[-h,0]$ for a constant $h\geqq 0$ and satisfying

$$\|\varphi\|=\sup\{|\varphi(\theta)|;\theta\in[-h,0]\}\leqq B^*.$$

We shall denote by $F^s(t,\varphi)$ the function defined by

$$F^s(t,\varphi)=F(t+s,\varphi).$$

Let $T(F)=\{F^s(t,\varphi);s\in I\}$ and let $H(F)$ be the closure of the set $T(F)$ in the sense of the uniform convergence on any compact subset of $I\times\bar{C}_{B^*}$. We shall assume that $H(F)$ is compact, that is, any sequence $\{F^{s_n}(t,\varphi)\}$ in $T(F)$ contains a subsequence which converges to an element of $H(F)$ uniformly on any compact subset of $I\times\bar{C}_{B^*}$. As is well known, if $F(t,\varphi)$ is almost periodic in t

uniformly for $\varphi\in\bar{C}_{B^*}$, $H(F)$ consists of the restriction of elements of the hull of $F(t,\varphi)$ to $[0,\infty)$ and $H(F)$ is compact.

Definition. Let $\xi(t)$ be a solution of the system (1) which satisfies $\|\xi_t\|\leqq B$, $B<B^*$, for all $t\geqq 0$. The solution $\xi(t)$ is said to be totally stable, if for any $t_0\geqq 0$ and any $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ such that if $G(t,\varphi)$ is continuous on $[t_0,\infty)\times\bar{C}_{B^*}$ and satisfies

$$|G(t,\varphi)-F(t,\varphi)|<\delta(\varepsilon) \quad \text{for all } (t,\varphi)\in[t_0,\infty)\times\bar{C}_{B^*}, \|\varphi-\xi_t\|\leqq\varepsilon,$$

and if $\psi\in\bar{C}_{B^*}$ satisfies

$$\|\psi-\xi_{t_0}\|<\delta(\varepsilon),$$

then any solution $x(t)$ through (t_0,ψ) of the system

$$\dot{x}(t)=G(t,x_t) \tag{2}$$

satisfies

$$\|\xi_t-x_t\|<\varepsilon \quad \text{for all } t\geqq t_0.$$

First of all, we shall prove the following lemma.

Lemma 1. *Let $\xi(t)$ be a solution of the system (1) which satisfies $\|\xi_t\|\leqq B$, $B<B^*$, for all $t\geqq 0$. Then $\xi(t)$ is totally stable if and only if for any $t_0\geqq 0$ and any $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ such that if $g(t)$ is continuous on $[t_0,\infty)$ and satisfies $|g(t)|<\delta(\varepsilon)$ for all $t\geqq t_0$ and if $\psi\in\bar{C}_{B^*}$ satisfies*

$$\|\psi-\xi_{t_0}\|<\delta(\varepsilon),$$

then any solution $y(t)$ through (t_0,ψ) of the system

$$\dot{x}(t)=F(t,x_t)+g(t) \tag{3}$$

satisfies

$$\|\xi_t-y_t\|<\varepsilon \quad \textit{for all } t\geqq t_0.$$

Proof. The necessity of the condition is clear. We shall now prove the sufficiency of the condition. Suppose that there exists a $t_0\in I$, a function $G(t,\varphi)$ and a solution $x(t)$ of the system (2) such that $\|x_{t_0}-\xi_{t_0}\|<\delta(\varepsilon)$ and $\|x_{t_1}-\xi_{t_1}\|=\varepsilon$ for some $t_1(>t_0)$, though $|G(t,\varphi)-F(t,\varphi)|<\delta(\varepsilon)$ for all $(t,\varphi)\in[t_0,\infty)\times\bar{C}_{B^*}$, $\|\varphi-\xi_t\|\leqq\varepsilon$, where $\delta(\varepsilon)$ is the one given in the condition. Here we can assume that $\varepsilon<B^*-B$ and $\|\xi_t-x_t\|\leqq\varepsilon$ for all t, $t_1\geqq t\geqq t_0$. If we set

$$g(t)=G(t,x_t)-F(t,x_t) \quad \text{for } t_1\geqq t\geqq t_0,$$

then $|g(t)|<\delta(\varepsilon)$ for $t_1\geqq t\geqq t_0$ and $g(t)$ is continuous on $[t_0,t_1]$. Moreover, $g(t)$ can easily be extended to the interval $[t_0,\infty)$ so that $|g(t)|<\delta(\varepsilon)$ for all $t\geqq t_0$.

Now consider the system (3). Then we can find a solution $y(t)$ of (3) such that $y(t)=x(t)$ for $t \leqq t_1$. Obviously, $\|\xi_{t_0}-y_{t_0}\|=\|\xi_{t_0}-x_{t_0}\|<\delta(\varepsilon)$ and $\|\xi_{t_1}-y_{t_1}\|=\varepsilon$. Thus there arises a contradiction. This proves the lemma.

To discuss the total stability of a given solution, by Lemma 1, it is sufficient to consider the system (3) instead of (2).

Now we shall prove the following lemma, which is a generalization of Lemma 6 in [3] and will be proved by the same idea as in the proof of Lemma 6 in [3].

Lemma 2. *Suppose that $H(F)$ is compact and for every $G \in H(F)$ the solution of (2) is unique for the initial condition. Let $T>0$, $B_1(<B^*)$ and K^* be given, where K^* is a compact subset of $\bar{C}_{B^*}$. Then, for any $\varepsilon>0$, there exists a $\delta(\varepsilon)>0$ such that for any $t_0 \in I$, if $x(t)$ is a solution of the system (1) which satisfies $x_{t_0} \in K^*$ and $\|x_t\| \leqq B_1$ for all $t_0 \leqq t \leqq t_0+T$, and if $g(t)$ is a continuous function such that $|g(t)|<\delta(\varepsilon)$ on $[t_0, t_0+T]$, we have*

$$\|x_t-y_t\|<\varepsilon \quad \text{for all } t \in [t_0, t_0+T],$$

whenever $y(t)$ is a solution of the system (3) satisfying $\|x_{t_0}-y_{t_0}\|<\delta(\varepsilon)$.

Proof. Suppose that there is no δ which satisfies the conditions in this lemma. Then, for some $\varepsilon>0$, $\varepsilon<B^*-B_1$, there exist sequences $\{\delta_k\}$, $\{t_k\}$, $t_k \in I$, $\{x^k(t)\}$, $\{g_k(t)\}$, $\{y^k(t)\}$ and $\{\tau_k\}$ such that $\delta_k \to 0$ as $k \to \infty$, $\|x^k{}_t\| \leqq B_1$, $x^k{}_{t_k} \in K^*$, $\|x^k{}_{t_k}-y^k{}_{t_k}\|<\delta_k$, $|g_k(t)|<\delta_k$, $\tau_k \in [t_k, t_k+T]$ and $\|x^k{}_{\tau_k}-y^k{}_{\tau_k}\|=\varepsilon$, where $x^k(t)$ is a solution of (1) and $y^k(t)$ is a solution of the system

$$\dot{x}(t)=F(t, x_t)+g_k(t).$$

Since $x^k{}_{t_k}$ are contained in the compact set K^* and $\tau_k-t_k \in [0, T]$, we can assume that $x^k{}_{t_k}$ converges to $\varphi^0 \in K^*$ and $\sigma_k=\tau_k-t_k$ to $\sigma \in [0, T]$ as $k \to \infty$. Moreover, clearly $y^k{}_{t_k}$ converges to φ^0 as $k \to \infty$, and hence $y^k{}_{t_k}$ belongs to a compact subset K_1 of $\bar{C}_{B^*}$.

Set $\xi^k(t)=x^k(t+t_k)$ for $t \in [-h, T]$, and let

$$\eta^k(t)=\begin{cases} y^k(t+t_k) & \text{for } t \in [-h, \sigma_k], \\ \eta^k(\sigma_k)=y^k(\tau_k) & \text{for } t \in [\sigma_k, T]. \end{cases}$$

Then $\xi^k(t)$ and $\eta^k(t)$ are solutions of

$$\dot{x}(t)=F(t+t_k, x_t) \quad \text{on } [0, T]$$

and

$$\dot{x}(t)=F(t+t_k, x_t)+g_k(t+t_k) \quad \text{on } [0, \sigma_k]$$

such that

$$\xi^k{}_0=x^k{}_{t_k} \text{ and } \eta^k{}_0=y^k{}_{t_k},$$

respectively. Since $\xi^k{}_0 \in K^*, \eta^k{}_0 \in K_1$, and since $\{\xi^k(t)\}$ and $\{\eta^k(t)\}$ are uniformly bounded and equicontinuous on $[0, T]$, there exists a compact subset S of $\bar{C}_{B^*}$ such that $\xi^k{}_t \in S$ and $\eta^k{}_t \in S$ for all $t \in [0, T]$ and all k. We can assume that $\xi^k(t)$ and $\eta^k(t)$ converge to $\xi(t)$ and $\eta(t)$ as $k \to \infty$, respectively, and that $F(t+t_k, \varphi)$ converges to $G(t, \varphi) \in H(F)$ uniformly on $[0, T] \times S$. Then clearly $\xi(t)$ is a solution through $(0, \varphi^0)$ of the system (2) on $[0, T]$, and $\eta(t)$ also is a solution through $(0, \varphi^0)$ of (2) on $[0, \sigma]$, because $|g_k(t+t_k)| < \delta_k$ and $\delta_k \to 0$ as $k \to \infty$. Therefore, because of the uniqueness, $\xi(t) \equiv \eta(t)$ on $[0, \sigma]$. However, $\|x^k{}_{\tau_k} - y^k{}_{\tau_k}\| = \varepsilon$, that is, $\|\xi^k{}_{\sigma_k} - \eta^k{}_{\sigma_k}\| = \varepsilon$ implies that $\|\xi_\sigma - \eta_\sigma\| = \varepsilon$. Thus there arises a contradiction. This proves the lemma.

By Lemmas 1 and 2, we have the following theorem.

Theorem. *For the system* (1) *we assume that* $H(F)$ *is compact and for each* $G \in H(F)$ *the solution of* (2) *is unique to the right for the initial function and that the system* (1) *has a solution* $\xi(t)$ *defined on* $[0, \infty)$ *which satisfies* $\|\xi_t\| \leqq B$ *for all* $t \geqq 0$ *and some* $B < B^*$ *and which is uniformly asymptotically stable for* $t \geqq 0$. *Then* $\xi(t)$ *is totally stable for* $t \geqq 0$.

Proof. Since $\xi(t)$ is uniformly asymptotically stable for $t \geqq 0$, for any $\varepsilon > 0$ there exists a $\delta_1(\varepsilon) > 0$ and a $T(\varepsilon) > 0$ such that for a solution $x(t)$ of the system (1) satisfying $\|x_{t_1} - \xi_{t_1}\| < \delta_1(\varepsilon)$ at some $t_1 \geqq 0$, we have $\|x_t - \xi_t\| < \varepsilon/2$ for all $t \geqq t_1$ and $\|x_t - \xi_t\| < \delta_1(\varepsilon)/2$ for all $t \geqq t_1 + T(\varepsilon)$. Here we can assume that $T(\varepsilon) > h$ and $\varepsilon < B^* - B$.

Since the initial function of $\xi(t)$ is continuous on the compact interval $[-h, 0]$ and $\xi(t)$ satisfies $|\xi(t')| \leqq B$ and $|\xi(t') - \xi(t'')| \leqq L|t' - t''|$ for all $t', t'' \geqq 0$, $\{\xi_t; t \geqq 0\}$ is contained in a compact subset K_1 of $\bar{C}_{B^*}$. Set $K^* = K \cup K_1$, where K is the compact subset of $\bar{C}_{B^*}$ consisting of $\varphi \in \bar{C}_{B^*}$ such that

$$|\varphi(\theta') - \varphi(\theta'')| \leqq L^* |\theta' - \theta''|$$

for all $\theta', \theta'' \in [-h, 0]$ and for a fixed constant $L^* > L$.

By Lemma 2, for $K^*, B_1 = (B^* + B)/2$, $\delta_1(\varepsilon)/2$ and $T(\varepsilon)$, there exists a $\delta(\varepsilon) > 0$, $\delta(\varepsilon) \leqq \min\{\delta_1(\varepsilon)/2, L^* - L\}$, such that for any $s \geqq 0$, if $x(t)$ is a solution of (1) defined on $s \leqq t \leqq s + T(\varepsilon)$ which satisfies $x_s \in K^*$ and $\|x_t\| \leqq B_1$ and if $\|x_s - \varphi\| < \delta(\varepsilon)$ and $|g(t)| < \delta(\varepsilon)$, then a solution $y(t)$ through (s, φ) of the system (3) exists on $[s, s + T(\varepsilon)]$ and satisfies

$$\|x_t - y_t\| < \frac{\delta_1(\varepsilon)}{2} \quad \text{on } [s, s + T(\varepsilon)].$$

For a fixed $t_0 \geqq 0$ consider a system (3), where $|g(t)| < \delta(\varepsilon)$ for all $t \geqq t_0$, and a solution $y(t)$ of (3) such that $\|\xi_{t_0} - y_{t_0}\| < \delta(\varepsilon)$. Since $\xi_{t_0} \in K^*$ and $\delta_1(\varepsilon) < \varepsilon$, we have

$$\|\xi_t - y_t\| < \frac{\delta_1(\varepsilon)}{2} < \varepsilon \quad \text{on } [t_0, t_0 + T(\varepsilon)]$$

and

$$\|\xi_{t_0+T(\varepsilon)} - y_{t_0+T(\varepsilon)}\| < \frac{\delta_1(\varepsilon)}{2}$$

by the mentioned above. From this and the fact that $T(\varepsilon) > h$ and $\delta_1(\varepsilon) < B^* - B$, it follows that $\|y_{t_0+T(\varepsilon)}\| \leqq B_1$ and $|\dot{y}(t)| \leqq L + \delta(\varepsilon) \leqq L^*$ on $t_0 + T(\varepsilon) - h \leqq t \leqq t_0 + T(\varepsilon)$, which implies that $y_{t_0+T(\varepsilon)} \in K \subset K^*$. Let $x(t)$ be a solution of (1) through $(t_0 + T(\varepsilon), y_{t_0+T(\varepsilon)})$. Since

$$\|\xi_{t_0+T(\varepsilon)} - x_{t_0+T(\varepsilon)}\| = \|\xi_{t_0+T(\varepsilon)} - y_{t_0+T(\varepsilon)}\| < \delta_1(\varepsilon),$$

we have

$$\|\xi_t - x_t\| < \frac{\varepsilon}{2} \quad \text{for all } t \geqq t_0 + T(\varepsilon)$$

and

$$\|\xi_{t_0+2T(\varepsilon)} - x_{t_0+2T(\varepsilon)}\| < \frac{\delta_1(\varepsilon)}{2}.$$

On the other hand, we have

$$\|x_t - y_t\| < \frac{\delta_1(\varepsilon)}{2} \quad \text{for all } t \in [t_0 + T(\varepsilon), t_0 + 2T(\varepsilon)],$$

because $\|x_{t_0+T(\varepsilon)} - y_{t_0+T(\varepsilon)}\| = 0$, $x_{t_0+T(\varepsilon)} \in K^*$, $|g(t)| < \delta(\varepsilon)$ and $\|x_t\| \leqq \|\xi_t\| + \varepsilon/2 \leqq B_1$ for all $t \geqq t_0 + T(\varepsilon)$. Thus we have

$$\|\xi_t - y_t\| < \varepsilon \quad \text{on } [t_0 + T(\varepsilon), t_0 + 2T(\varepsilon)]$$

and

$$\|\xi_{t_0+2T(\varepsilon)} - y_{t_0+2T(\varepsilon)}\| < \delta_1(\varepsilon).$$

By repeating the same argument, we have

$$\|\xi_t - y_t\| < \varepsilon \quad \text{on } t_0 + pT(\varepsilon) \leqq t \leqq t_0 + (p+1)T(\varepsilon), \ p = 2, 3, \cdots,$$

and hence we have

$$\|\xi_t - y_t\| < \varepsilon \quad \text{for all } t \geqq t_0,$$

whenever $t_0 \geqq 0$, $\|\xi_{t_0} - y_{t_0}\| < \delta(\varepsilon)$ and $|g(t)| < \delta(\varepsilon)$ for all $t \geqq t_0$. This completes the proof by Lemma 1.

References

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