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# On a Compact Invariant Set Isolated from Minimal Sets

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#### 1. Introduction and preliminaries.

Let a dynamical system  $(X, R, \pi)$  be given where:

1) X is a locally compact metric space,

2) R is the group of real numbers,

3)  $\pi$  is a continuous map of  $X \times R$  onto X:

$$(x,t) \xrightarrow{x} \pi(x,t), x \in X, t \in R, \pi(x,t) \in X,$$

such that

(i)  $\pi(x,0) = x$ ,

(ii)  $\pi(\pi(x,s),t) = \pi(x,s+t),$ 

#### $x \in X$ , $s, t \in R$ .

The following notation will be used throughout the paper:

(1)  $C^+(x)$  is the positive half-orbit from  $x \in X$ ,

(2)  $C^{-}(x)$  is the negative half-orbit from  $x \in X$ ,

 $(3) \quad C(x) = C^+(x) \cup C^-(x),$ 

(4)  $L^+(x)$  is the  $\omega$ -limit set of x,

(5)  $L^{-}(x)$  is the  $\alpha$ -limit set of x,

(6) 
$$J^+(x)$$
 is the positive prolongational limit set<sup>1</sup>) of x,

(7)  $J^{-}(x)$  is the negative prolongational limit set of x.

Let F be a compact invariant set of our dynamical system and U be a neighbourhood of F. The purpose of the present paper is to investigate the behaviour of orbits passing through  $\overline{U}-F$  and thereby to characterize the nature of F. For that purpose, we divide  $\overline{U}-F$  into following subsets:

$$G_U = [x; x \in \overline{U} - F, C^+(x) \oplus \overline{U}, C^-(x) \oplus \overline{U}],$$
  
 $N_U^+ = [x; x \in \overline{U} - F, C^+(x) \subset \overline{U}],$   
 $N_U^- = [x; x \in \overline{U} - F, C^-(x) \subset \overline{U}],$   
 $N_U = N_U^+ \cap N_U^-,$ 

so that

<sup>1)</sup> For the definition of the prolongational limit set, see [2], p.122.

$$U-F=G_U \cup N_U^{\dagger} \cup N_U^{\dagger},$$
  
$$G_U \cap N_U^{\dagger}=\phi, \ G_U \cap N_U^{\dagger}=\phi.$$

As is evident from the definition,  $G_U$  is open and  $N_U^+$ ,  $N_U^-$  and  $N_U$  are closed in  $\overline{U}-F$ .

The idea of dividing  $\overline{U}-F$  into these subsets has its origin in Bendixson's famous memoir<sup>2)</sup>, where he used it successfully for the study of isolated critical points of a dynamical system in  $S^2$ . The above definition is its natural extension. In my previous paper<sup>3)</sup>, I introduced this definition for the purpose of studying the isolated compact minimal sets. However the same idea works for the study of compact invariant sets as well, and, in certain points at least, the assumption of minimality adopted in my paper seems to be rather too stringent and sometimes even superfluous. Indeed, some of the results obtained there admit immediate generalization to the case when F is a compact invariant set not necessarily minimal after a slight modification. So let us begin with the generalization of some of those theorems and then use them for the further study of compact invariant sets.

#### 2. Fundamental theorems on a saddle set.

Throughout the paper we assume that F is a compact and non-open invariant set isolated from minimal sets. Here the isolatedness is defined as follows:

**Definition 1.** F is said to be *isolated* from minimal sets if and only if there exists a neighbourhood U of F such that any minimal set contained in U is a subset of F.

This assumption, together with the local compactness of X, will ensure the existence of a compact neighbourhood D of F such that any minimal set contained in D is a subset of F. As neighbourhoods of F for which the sets G,  $N^+$ ,  $N^-$  and N are considered, we consider only those contained in D. So, for any neighbourhood U of F considered here, we assume that

(i) U is relatively compact, and

(ii)  $\overline{U}$  contains no minimal sets except those contained in F, without any further notice.

From those assumptions on U, it follows immediately that any closed invariant set in  $\overline{U}$  intersects F. Indeed, if M is such a closed invariant set, M is compact by (i). So M contains at least one minimal set which should necessarily be a subset of F by (ii). Hence  $M \cap F \neq \phi$ .

From this remark, we obtain:

2)	[1].	

3) [3].

**Theorem 1.** (1) If  $x \in N_U^+, L^+(x) \cap F \neq \phi$ . (2) If  $x \in N_U^-, L^-(x) \cap F \neq \phi$ .

In our terminology, the definition of a saddle set, introduced by Seibert and Ura<sup>4</sup>), can be stated as follows:

**Definition 2.** F is called a saddle set if there exists a neighbourhood U of F such that  $\overline{G}_U \cap F \neq \phi$ , or equivalently, if  $\overline{G}_U \cap F \neq \phi$  for every sufficiently small nieghbourhood U of F.

The following Theorem 2 and Theorem 3 are the generalization of Theorem. 8 and Theorem 9 in [3] respectively.

**Theorem 2.** If F is a saddle set, then (1) there exists  $x \in X - F$  such that

$$L^+(x) \cap F \neq \phi, J^+(x) \cap (X-F) \neq \phi,$$

and

(2) there exists  $x' \in X - F$  such that

$$L^{-}(x') \cap F \neq \phi, J^{-}(x') \cap (X-F) \neq \phi.$$

Conversely if (1) or (2) holds, then F is a saddle set.

**Proof.** First suppose that F is a saddle set. Then, by Definition 2, there exists a neighbourhood U of F such that

 $\overline{G}_U \cap F \neq \phi$ .

So we can find  $\{y_n\} \subset G_{II}$  with  $y_n \to y \in F$ . As  $C^+(y_n) \oplus \overline{U}$  and  $C^-(y_n) \oplus \overline{U}^-$ , there exist  $t_n < 0$  and  $t'_n > 0$  such that

$$\pi(y_n, t) \in U, \quad t_n < t < t'_n,$$
  

$$\pi(y_n, t_n) \in \partial U,$$
  

$$\pi(y_n, t'_n) \in \partial U,$$

where the symbol  $\partial$  represents the boundary. As  $y_n \rightarrow y \in F$  and F is invariant, we have

$$t_n \to -\infty, \quad t'_n \to \infty \quad \text{as } n \to \infty.$$

Put  $x_n = \pi(y_n, t_n)$  and  $x'_n = \pi(y_n, t'_n)$ . Then  $\partial U$  being compact (assumption (i) on U), we may suppose that

$$x_n \to x \in \partial U, \ x'_n \to x' \in \partial U.$$

Then the routine continuity argument will show that  $x \in N_U^+$  and  $x' \in N_U^-$  and hence

4) See [4], p. 277.

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 $L^+(x) \cap F \neq \phi$ ,  $L^-(x') \cap F \neq \phi$ ,

by Theorem 1.

Now since

$$\pi(x_n,\tau_n) = x'_n, \ \pi(x'_n,-\tau_n) = x_n$$

where

 $\tau_n = t'_n - t_n \to \infty$  as  $n \to \infty$ ,

we have

$$x' \in J^+(x)$$
 and  $x \in J^-(x')$ .

Since  $x \in \partial U$  and  $x' \in \partial U$ , and  $\partial U \subset X - F$ ,

$$J^+(x) \cap (X-F) \neq \phi, J^-(x) \cap (X-F) \neq \phi.$$

Thus we have proved the existence of x and x' with required properties.

Conversely assume that (1) holds. Then we can find a neighbourhood U of F such that

 $\overline{U} \oplus x$ ,  $\overline{U} \oplus J^+(x)$ .

Let V be an arbitrary neighbourhood of F contained in U. Since  $L^+(x) \cap F \neq \phi$ ,  $C^+(x) \cap V \neq \phi$ . Let  $y \in C^+(x) \cap V$ . Then as  $x \in C^-(y)$  and  $x \notin \overline{U}$ ,

 $C^{-}(y) \subset \overline{U}.$ 

Since  $\overline{U} \Rightarrow J^+(x)$ , there exists a  $z \in J^+(x) \cap (X - \overline{U})$ . As  $y \in C^+(x), J^+(x) = J^+(y)$ . So  $z \in J^+(y)$ . Therefore there exist  $\{y_n\} \subset X$  and  $\{t_n\} \subset R$  such that

 $y_n \rightarrow y, t_n \rightarrow \infty, \pi(y_n, t_n) \rightarrow z.$ 

If *n* is large enough,  $y_n \in V$ , and also from (1),  $C^-(y_n) \oplus \overline{U}$ . Also as  $\pi(y_n, t_n) \to z$ and  $z \oplus \overline{U}, \pi(y_n, t_n) \oplus \overline{U}$  if *n* is sufficiently large which shows that  $C^+(y_n) \oplus \overline{U}$ . Thus, by choosing *n* large enough to satisfy all these conditions, we have found in *V* a point  $y_n$  with  $C^+(y_n) \oplus \overline{U}, C^-(y_n) \oplus \overline{U}$ . Therefore  $G_U \cap V \neq \phi$  for any neighbourhood *V* of *F* which means that *F* is a saddle set.

The proof is similar if (2) is assumed.

**Theorem 3.** If F is not a saddle set,

(1)  $x \in N_U^+$  implies  $L^+(x) \subset F$ , and

(2)  $x \in N_U$  implies  $L^-(x) \subset F$ .

**Proof.** We shall give the proof of (1) only.

If  $x \in N_U^+$ ,  $L^+(x) \cap F \neq \phi$  by Theorem 1.

If  $L^+(x) \oplus F$ ,  $L^+(x) \cap (X-F) \neq \phi$ . But as  $J^+(x) \supset L^+(x)$ ,  $J^+(x) \cap (X-F) \neq \phi$ .

 $\phi$ . Hence, by Theorem 2, F is a saddle set contrary to the assumption.

Examining the proof of Theorem 2, we immediately obtain:

**Theorem 4.** If F is a saddle set,

 $G_U \neq \phi$ ,  $N_U^+ \neq \phi$ ,  $N_U^- \neq \phi$ ,

if U is sufficiently small.

#### 3. Stability.

Here we shall give the criterion for stability of F in terms of the structure of  $\overline{U}-F$ . Following definitions of stability and asymptotic stability are well-known.

**Definition 3.** A compact invariant set M is said to be *positively* (negatively) stable if, for any neighbourhood U of M, there is a neighbourhood V of M such that  $x \in V$  implies  $C^+(x) \subset U$  ( $C^-(x) \subset U$ ).

**Definition 4.** A compact invariant set M is said to be *positively* (*negatively*) asymptotically stable if it is positively (negatively) stable and also there exists a neighbourhood V of M such that  $x \in V$  implies  $L^+(x) \subset M$  ( $L^-(x) \subset M$ ).

First let us prove:

**Theorem 5.** F is positively (negatively) asymptotically stable whenever it is positively (negatively) stable.

**Proof.** As is obvious from the definition, F is positively stable if and only if  $N_U^+ \cup F$  contains a neighbourhood of F for any U. Therefore if V is a sufficiently small neighbourhood of F, then  $N_U^+ \supset V - F$  and hence  $G_U \cap V = \phi$ . This being valid for any U, F is not a saddle set. So by Theorem 3,  $x \in V F \subset N_U^+$  implies  $L^+(x) \subset F$ . Hence F is positively asymptotically stable.

So, for a compact invariant set isolated from minimal sets, stability and asymptotic stability are equivalent concepts.

**Theorem 6.** (1) F is positively stable (and hence positively asymptotically stable) if and only if  $N_{\overline{U}}=\phi$  for some neighbourhood U of F.

(2) F is negatively stable (and hence negatively asymptotically stable) if and only if  $N_U^+=\phi$  for some neighbourhood U of F.

**Proof.** We shall give the proof of (1) only.

First let us notice that if  $N\overline{U}=\phi$  for some U, we have  $N\overline{V}=\phi$  for every neighbourhood V of F contained in U. Therefore the statement " $N\overline{U}=\phi$  for some U" is equivalent to the stronger one " $N\overline{U}=\phi$  for every sufficiently small U".

Suppose that  $N\overline{U}=\phi$  for any small U. Then by Theorem 4, F is not a saddle set. Hence for any U, there exists a neighbourhood V of F such that

$$V \subset U, \ G_U \cap V = \phi.$$

But since  $N\overline{U}=\phi$ , this means that  $N\overline{U}\supset V-F$  which implies the stability of F.

Conversely suppose that F is positively stable and  $N_U^{-} \neq \phi$  for some U. Let  $x \in N_U^{-}$  and W be a neighbourhood of F such that  $x \notin W$ . As  $x \in N_U^{-}$  implies  $L^{-}(x) \cap F \neq \phi$ ,  $C^{-}(x) \cap V \neq \phi$  for any small neighbourhood V of F. Let  $y \in C^{-}(x) \cap V$ . Then, as  $x \notin W$  and  $x \in C^{+}(y)$ , we have  $C^{+}(y) \notin W$ . Therefore F is not stable and we have come to a contradiction.

#### 4. Application to attractor theory.

**Definition 5.** A compact invariant set M is called a *positive (negative) attractor* if and only if there exists a neighbourhood V of M such that  $x \in V$ implies  $L^+(x) \neq \phi$  and  $L^+(x) \subset M$  ( $L^-(x) \neq \phi$  and  $L^-(x) \subset M$ ).

**Definition 6.** A compact invariant set M is called a *positive (negative) weak* attractor if and only if there exists a neighbourhood V of M such that  $x \in V$  implies  $L^+(x) \cap M \neq \phi$   $(L^-(x) \cap M \neq \phi)$ .

**Definition 7.** Let M be a compact invariant set. Then the set

$$A^+(M) = [x; x \in X, L^+(x) \neq \phi, L^+(x) \subset M]$$

is called the region of positive attraction of M, and the set

 $A^{-}(M) = [x; x \in X, L^{-}(x) \neq \phi, L^{-}(x) \subset M]$ 

is called the region of negative attraction of M.

**Definition 8.** Let M be a compact invariant set. Then the set

$$x^{+}(M) = [x; x \in X, L^{+}(x) \cap M \neq \phi]$$

is called the region of positive weak attraction of M, and the set

$$a^{-}(M) = [x; x \in X, L^{-}(x) \cap M \neq \phi]$$

is called the region of negative weak attraction of M.

Evidently  $A^+(M)$ ,  $A^-(M)$ ,  $a^+(M)$  and  $a^-(M)$  are all invariant sets, and M is contained in all of them. Also

$$a^+(M) \supset A^+(M), \quad a^-(M) \supset A^-(M).$$

M is a positive (negative) attractor if and only if  $A^+(M)$   $(A^-(M))$  contains a neighbourhood of M, and is a positive (negative) weak attractor if and only if  $a^+(M)$   $(a^-(M))$  contains a neighbourhood of M.

Lemma 1. If F is not a saddle set, we have

$$A^+(F) = a^+(F), A^-(F) = a^-(F).$$

**Proof.** Since  $a^+(F) \supset A^+(F)$ , we have only to show that

$$A^+(F) \supset a^+(F)$$
.

Let  $x \in a^+(F)$ . Then  $L^+(x) \cap F \neq \phi$  by definition. If  $L^+(x) \oplus F$ ,  $L^+(x) \cap$ 

 $(X-F) \neq \phi$ . As  $J^+(x) \supset L^+(x)$ , this implies  $J^+(x) \cap (X-F) \neq \phi$ . Since  $x \notin F$  is obvious, this shows, by Theorem 2, that F is a saddle set contrary to the assumption. Hence  $L^+(x) \subset F$  and we get

$$A^+(F) \supset a^+(F).$$

Similarly we can prove that  $A^{-}(F) = a^{-}(F)$ .

From Lemma 1, we immediately have:

**Theorem 7.** If F is not a saddle set, it is an attractor whenever it is a weak attractor.

For convenience, we write

$$A^{+}(F) - F = B^{+}(F), A^{-}(F) - F = B^{-}(F),$$
  
 $a^{+}(F) - F = b^{+}(F), a^{-}(F) - F = b^{-}(F).$ 

Then from Lemma 1, we have

$$B^+(F) = b^+(F), B^-(F) = b^-(F)$$

whenever F is not a saddle set.

To prepare for the proof of Theorem 8 which will be mentioned in the next section, we now prove the following Lemmas 2, 3 and 4. Although the Theorem 8 itself is concerned with compact phase space, those lemmas are valid for non-compact phase space. So we shall prove them without assuming the compactness of X.

**Lemma 2.** If F is not a saddle set,  $B^+(F) = b^+(F)$  and  $B^-(F) = b^-(F)$  are both open.

**Proof.** As X-F is open, it is sufficient to show that  $B^+(F)$  and  $B^-(F)$  are open in X-F. We shall give the proof for  $B^+(F)$  only.

As the lemma is trivial when  $B^+(F) = \hat{\phi}$  or  $B^+(F) = X - F$ , we suppose that

$$B^+(F) = \phi, B^+(F) = X - F.$$

Assume that  $B^+(F)$  is not open in X-F. Then there exists  $\{x_n\} \subset B^+(F)$  $\cup F = A^+(F)$  such that  $x_n \to x \in B^+(F)$ . Since  $x_n \in A^+(F) = a^+(F)$ ,  $L^+(x_n) \cap F = \phi$ . Therefore if U is an open neighbourhood of F which contains no minimal sets except those contained in F,

$$L^+(x_n) \cap (X-U) \neq \phi$$

for every *n*. Indeed, if not,  $L^+(x_n) \subset U \subset \overline{U}$  and this implies  $L^+(x_n) \cap F \neq \phi$  as we have remarked in section 2.

Hence positive numbers  $t_n > n$  can be found so that

$$\pi(x_n, t_n) \in X - U.$$

Let  $\widetilde{X}$  be a one-point compactification of X and  $(\widetilde{X}, R, \widetilde{\pi})$  be the extension

of  $(X, R, \pi)$  onto  $\widetilde{X}$ . Also let us denote by  $\widetilde{J}^+(x)$  the positive prolongational limit set of x in this extended dynamical system. Obviously  $\widetilde{J}^+(x) \cap X = J^+(x)$ . Then as

$$\pi(x_n,t_n) \in X - U \subset \widetilde{X} - U$$

and  $\widetilde{X}-U$  is compact,  $\{\pi(x_n, t_n)\} = \{\widetilde{\pi}(x_n, t_n)\}$  has a cluster point y in  $\widetilde{X}-U$ . Since  $t_n \to \infty$ ,  $x_n \to x, y$  belongs to  $\widetilde{J}^+(x)$ .

If  $y \in X$ , then  $y \in J^+(x)$ , and as  $y \in X - U$ ,

$$J^+(x) \cap (X-F) \neq \phi.$$

Also as  $x \in B^+(F)$ , we have  $L^+(x) \cap F \neq \phi$ . Thus by Theorem 2, F must be a saddle set contrary to the assumption.

If  $y \in \widetilde{X}$ , that is, y is the point at infinity, then  $y \notin J^+(x)$ . However it is known that if the phase space is locally compact, prolongational limit set is connected whenever it is compact<sup>5)</sup>. Now  $\widetilde{X}$  being compact,  $\widetilde{J}^+(x)$  is a connected set intersecting both F and the point at infinity. Therefore  $\widetilde{J}^+(x)$  must contain a point in X-U, which means, by the same argument as above, that F is a saddle set. Thus we have again come to a contradiction and therefore  $B^+(F)$  must be open.

**Lemma 3.** If F is not a saddle set,  $A^+(F) \cup A^-(F)$  is an open set containing F.

**Proof.** Let U be a neighbourhood of F. As F is not a saddle set, an open neighbourhood V of F can be found so that

$$V \subset \overline{U}, \ G_U \cap V = \phi.$$

So, if we put

$$N_{U}^{+} \cap V = N^{+}, N_{U}^{-} \cap V = N^{-},$$

we have

$$V - F = N^+ \cup N^-$$
.

From Theorem 1 and Lemma 1, we have

$$A^+(F) = a^+(F) \supset N_U^+ \supset N^+,$$
  
$$A^-(F) = a^-(F) \supset N_U^- \supset N^-.$$

Also we know that

 $A^+(F) \supset F, A^-(F) \supset F.$ 

Consequently

5) Cf. [2], p. 123.

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### $A^+(F) \cup A^-(F) \supset N^+ \cup N^- \cup F = V.$

To show that  $A^+(F) \cup A^-(F)$  is open, we have only to notice that

$$A^+(F) \cup A^-(F) = B^+(F) \cup B^-(F) \cup F,$$
  
$$A^+(F) \cup A^-(F) \supset V \supset F,$$

and hence

$$A^{+}(F) \cup A^{-}(F) = B^{+}(F) \cup B^{-}(F) \cup V.$$

Since V is open and  $B^+(F)$  and  $B^-(F)$  are open by Lemma 2,  $A^+(F) \cup A^-(F)$  is open.

**Lemma 4.** If F is not a saddle set and there exists a fundamental system of neighbourhoods V(F) of F such that V-F is connected for every  $V \in V(F)$ , then  $B^+(F) \cup B^-(F)$  is connected.

**Proof.** If not, there exist two sets M and N both non-empty and open in  $B^+(F) \cup B^-(F)$  such that

$$B^+(F) \cup B^-(F) = M \cup N, M \cap N = \phi.$$

Since  $A^+(F) \cup A^-(F)$  contains a neighbourhood of F by Lemma 3, there is a  $V \in V(F)$  such that

$$B^+(F) \cup B^-(F) \supset V - F.$$

So, V-F being connected by assumption, we have either  $V-F \subset M$  or  $V-F \subset N$ .

If  $V-F \subset M$ , then  $N \subset X-V$  and therefore  $\overline{N} \cap F = \phi$ . As N is a connected component (or a union of connected components) of an invariant set  $B^+(F) \cup$  $B^-(F)$ , it is itself an invariant set. Therefore  $x \in N$  implies  $\overline{C(x)} \subset \overline{N}$  and hence  $\overline{C(x)} \cap F = \phi$ . This is however a contradiction since  $x \in N$  implies

$$x \in B^+(F) \cup B^-(F) \subset A^+(F) \cup A^-(F)$$

and hence  $L^+(x) \cap F \neq \phi$  or  $L^-(x) \cap F = \phi$ . So we must have

 $V - F \subset M$ .

Similarly we get  $V - F \subset N$ . Thus we have come to a contradiction.

#### 5. The case of a compact phase space.

**Theorem 8.** Let X be a compact metric space and F be a compact and nonopen invariant set isolated from minimal sets. If:

(1) F is not a saddle set,

(2) there exists a fundamental system of neighbourhoods V(F) of F such that V-F is connected for every  $V \in V(F)$ , and

 $\langle (3) \rangle$  F is not an attractor,

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then  $A^+(F) \cup A^-(F)$  is an open set containing F with non-empty boundary and at least one saddle minimal set is contained in this boundary.

**Proof.** From Lemma 3, we already know that  $A^+(F) \cup A^-(F)$  is an open set containing F.

As F is not an attractor, it implies

$$A^+(F) \neq A^+(F) \cup A^-(F), A^-(F) \neq A^+(F) \cup A^-(F).$$

From this we immediately have

$$B^+(F) \neq \phi, B^-(F) \neq \phi, B^+(F) \neq B^-(F).$$

If we notice that  $B^+(F)$  and  $B^-(F)$  are both open by Lemma 2 and  $B^+(F) \cup B^-(F)$  is connected by Lemma 4, the first two relations imply  $W=B^+(F)\cap B^-(F) \neq \phi$  and the third relation implies that  $K=B^+(F)\cup B^-(F)-W \neq \phi$ .

As W is open, K is closed in  $B^+(F) \cup B^-(F) = K \cup W$ . So there exists  $\{x_n\} \subset W$  such that  $x_n \to x \in K$ .

Suppose, for example,  $x \in B^+(F)$  and  $x \in B^-(F)$ . Since X is compact,  $L^-(x) \neq \phi$ .

As  $B^{-}(F) = b^{-}(F)$  by assumption (i) and Lemma 1,

$$L^{-}(x) \cap F = \phi.$$

Also, as  $B^+(F) \cup B^-(F)$  is invariant,

$$L^{-}(x) \subset \overline{B^{+}(F) \cup B^{-}(F)}.$$

But, as is evident from the definition,  $B^+(F) \cup B^-(F)$  contains no compact invariant sets at all. Therefore

$$L^{-}(x) \subset \partial(B^{+}(F) \cup B^{-}(F)) - F$$
  
=  $\partial(A^{+}(F) \cup A^{-}(F)).$ 

Hence  $\partial(A^+(F) \cup A^-(F))$  is non-empty.

Let F' be a minimal set in  $L^{-}(x)$ . We shall show that F' is a saddle set.

Let V be a neighbourhood of F' with  $\overline{V} \cap F = \phi$ , and S be an arbitrary neighbourhood of F' contained in V. As  $L^{-}(x) \supset F'$ ,  $C^{-}(x)$  intersects with S. Then, as  $x_n \rightarrow x$ ,  $C^{-}(x_n)$  also intersects with S if n is large. Let  $y \in C^{-}(x_n) \cap S$ .

As  $x_n \in W$ , we have

$$L^+(y) = L^+(x_n) \subset F, \ L^-(y) = L^-(x_n) \subset F,$$

and hence

 $C^+(y) \subset \overline{V}, \ C^-(y) \subset \overline{V}.$ 

this shows that F' is a saddle set.

From Theorem 8, we can easily derive the following:

**Theorem 9.** Suppose that X is a compact metric space, and none of the minimal sets of  $(X, R, \pi)$  is a saddle set. Then a compact invariant set F is an attractor if:

(1) there exists a neighbourhood U of F such that U contains only a finite number of minimal sets, and

(2) there exists a fundamental system of neighbourhoods V(F) of F such that V-F is connected for every  $V \in V(F)$ .

**Proof.** As the theorem is trivial when F is open, we may suppose that it is not open.

From assumption (1), F is isolated from minimal sets. So, due to Theorem 8, we have only to show that F is not a saddle set.

Suppose that F is a saddle set. Then, by Theorem 2, there exists  $x \in X-F$  such that

$$L^+(x) \cap F \neq \phi$$
,  $J^+(x) \cap (X-F) \neq \phi$ .

Let M be a minimal set in  $L^+(x) \cap F$ . Then evidently

$$L^+(x) \cap M \neq \phi, J^+(x) \cap (X-M) \neq \phi.$$

Also, from assumption (1), M is isolated from other minimal sets. Thus, by Theorem 2, M is a saddle set contrary to the assumption.

**Corollary.** Suppose that X is a compact metric space and  $(X, R, \pi)$  has only a finite number of minimal sets and also none of those minimal sets is a saddle set. Then a compact invariant set F is an attractor if there exists a fundamental system of neighbourhoods V(F) of F such that V-F is connected for every  $V \in V(F)$ .

#### References

- [1] I. Bendixson, Sur les courbes définies par des équations différentielles, Acta Math. 24 (1901), 1-88.
- [2] N.P. Bhatia and G.P. Szegö, Dynamical systems: stability theory and applications, Lecture notes in mathematics, Springer (1967).
- [3] T. Saito, Isolated minimal sets, Funkcial. Ekvac. 11 (1968), 155-167.
- [4] T. Ura, On the flow outside a closed invariant set; stability, relative stability and saddle sets, Contrib. to Diff. Eqs. 3 (1964), 249-294.

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