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Derivation Pairs on the Holomorphic Functions

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Dedicated to the memory of Sundaram Seshu

Let $H(G)$ be the algebra of holomorphic functions on a region G in the complex plane, under the topology of uniform convergence on compact subsets of G. We use the term "derivation pair" to denote two continuous linear functionals on this algebra that satisfy, for all pairs f, g of elements of $H(G)$, the relation

$$L(fg)=L(f)M(g)+M(f)L(g). \tag{1}$$

Each point $\alpha\in G$ gives rise to the obvious derivation pair $L(f)=f'(\alpha)$, $M(f)=f(\alpha)$, but there are other derivation pairs than these. We find here all derivation pairs on $H(G)$. To begin with, we tacitly ignore the trivial case $L=0$.

By the representation theorem ([1], [2], [3]) for continuous linear functionals on $H(G)$, we may write

$$L(f)=\frac{1}{2\pi_i}\int_{\Gamma}f(z)l(z)dz$$

$$M(f)=\frac{1}{2\pi_i}\int_{\Gamma}f(z)m(z)dz$$

where

$$l(w)=L\left(\frac{1}{w-z}\right), \qquad m(w)=M\left(\frac{1}{w-z}\right)$$

and Γ is a closed rectifiable curve in G that has winding number 1 around each singular point of l or of m. It is easily seen that l and m are holomorphic functions in the closed complement of G on the Riemann sphere, and that $l(\infty)=m(\infty)=0$. Using (1) and the identity

$$\frac{1}{w_1-z}\frac{1}{w_2-z}=\frac{1}{w_1-w_2}\left(\frac{1}{w_2-z}-\frac{1}{w_1-z}\right)$$

we get

$$\frac{1}{w_1-w_2}\{l(w_2)-l(w_1)\}=l(w_1)m(w_2)+l(w_2)m(w_1). \tag{2}$$

On letting w_1 approach $w=w_2$, we get

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$$l'(w) = -2\,l(w)m(w).$$

Applying this in (2), we get

$$\frac{1}{w_1 - w_2}\{l(w_1) - l(w_2)\} = \frac{l(w_1)}{2}\frac{l'(w_2)}{l(w_2)} + \frac{l(w_2)}{2}\frac{l'(w_1)}{l(w_1)},$$

or

(3) $$l'(w_1)(w_1 - w_2)l(w_2)^2 + l(w_1)^2[l'(w_2)(w_1 - w_2) - 2l(w_2)] + l(w_1)\,2\,l(w_2)^2 = 0.$$

Similarly, for any w_3 in the complement of G, we have

(4) $$l'(w_1)(w_1 - w_3)l(w_3)^2 + l(w_1)^2[l'(w_3)(w_1 - w_3) - 2l(w_3)] + l(w_1)\,2\,l(w_3)^2 = 0.$$

Eliminating $l'(w_1)$ from these equations, and setting $w = w_1$, we see that $l(w)$ has the form

(5) $$l(w) = \frac{1}{Aw^2 + Bw + C}$$

for suitable constants A, B, C (that depend on w_2 and w_3). Since $l(\infty) = 0$, we cannot have both $A = 0$ and $B = 0$. Therefore (5) has one of the forms

(6) $$l(w) = D\left(\frac{1}{w - \alpha} - \frac{1}{w - \beta}\right)$$

$$l(w) = D\frac{1}{(w - \alpha)^2}$$

$$l(w) = D\frac{1}{w - \alpha}.$$

In each of these cases, we solve for $m(w)$ to get the corresponding representations

(6′) $$m(w) = \frac{1}{2}\left(\frac{1}{w - \alpha} + \frac{1}{w - \beta}\right)$$

(7′) $$m(w) = \frac{1}{w - \alpha}$$

(8′) $$m(w) = \frac{1}{2}\frac{1}{w - \alpha}.$$

Therefore, we have as the only possible derivation pairs, where E denotes a non-zero constant

(6″) $$L(f) = E(f(\alpha) - f(\beta)), \qquad M(f) = \frac{1}{2}(f(\alpha) + f(\beta))$$

(7″) $$L(f) = Ef'(\alpha), \qquad M(f) = f(\alpha)$$

(8″) $$L(f) = Ef(\alpha), \qquad M(f) = \frac{1}{2}f(\alpha).$$

Since it is easy to check that each of (6″), (7″), (8″) is a derivation pair, we have completely solved the problem.

The corresponding problem is open when G is an arbitrary Riemann surface, and for functions of several complex variables. Finally, we ask whether the relation (1) between two linear functionals L and M, $L\neq 0$, implies that they must be continuous.

References

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