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L^p-stability of Non-Linear Differential-Difference Equations

By D. Ramakrishna RAO

(University Nehru Nagar)

L^p-stability of ordinary differential equations has been first studied by Aaron Strauss, in his doctoral dissertation [1]. We extend his results to non-linear differential-difference equations. We consider the non-linear differential-dfference equations

(1) $$x_t^1(0)=f(x_t, t)$$

(2) $$y_t^1(0)=g(y_t, t)$$

and study the L^p-stability of the systems. In this paper we use the following notations.

R^n is the space of n vectors and for $x\in R^n$, $||x||$ is any vector norm. Given a number $h>0$, we can find c which denotes the space of continuous functions mapping the interval $[-h, 0]$ into R^n and for $\varphi\in c$, $||\varphi||=\sup_{-h\leqq\theta\leqq 0}||\varphi(\theta)||$, c_H will denote the set of $\varphi\in c$ for which $||\varphi||\leqq H$. We use the symbol $||\ \ ||$ to denote the norm in whatever space under consideration. For any continuous function $y(u)$ defined on $-h\leqq u\leqq A$, $A\geqq 0$, and any fixed t, $0\leqq t\leqq A$, the symbol y_t will denote the function $y(t+\theta)$, $-h\leqq\theta\leqq 0$ i. e. $y_t\in c$ and is the segment of the function $y(u)$ by letting u range in the interval $t-h\leqq u\leqq t$.

Let $f(\varphi, t)$ and $g(\psi, t)$ be non-linear in φ and ψ respectively and is continuous in t, φ and in t, ψ for all $t\geqq 0$ and φ, $\psi\in c_H$. Let $x_t^1(0)$ denote the right hand derivative of the function $x(u)$ at $u=t$. $f(\varphi, t)$ and $g(\psi, t)$ are Lipschitzian in φ and ψ with Lipschitz constant L.

Let $t_0\geqq 0$ and let $\varphi\in c_H$ be any given function. A function $x_t(t_0, \varphi)$ is said to be a solution of (1) with initial function φ at time t_0, if there exists an $A>0$ such that

(i) for each t, $t_0\leqq t\leqq t_0+A$, $x_t(t_0, \varphi)$ is defined and $\in c_H$.

(ii) $x_{t_0}(t_0, \varphi)=\varphi$.

(iii) $x_t^1(0)=f(x_t, t)$, $t_0\leqq t\leqq t_0+A$.

Similar definition can be given for the solution $y_t(t_0, \psi)$ of the equation (2).

Let $V(t, \varphi, \psi)$ be a continuous function in t, φ, and ψ for $t\geqq 0$, $\varphi, \psi\in c_H$. The derivative of V along the solutions of (1) and (2) will be denoted by $\dot{V}_{(1),(2)}$ and is defined as

(3) $$\dot{V}_{(1),(2)}(t,\varphi,\psi) = \overline{\lim_{h\to 0+}} \frac{1}{h}[V(t+h, x_{t+h}(t,\varphi), y_{t+h}(t,\psi)) - V(t,\varphi,\psi)],$$

where φ and ψ are the solutions of (1) and (2) respectively at time t_0.

Lemma 1. *Let $f(\varphi,t)$ and $g(\psi,t)$ satisfy the above conditions. Let the solutions $x_t(t_0,\varphi)$ and $y_t(t_0,\psi)$ of (1) and (2) respectively satisfy the condition*

(4) $$\|x_t(t_0,\varphi) - y_t(t_0,\psi)\|^{(h)} \leqq K(t_0)e^{-[\alpha(t)-\alpha(t_0)]}\|\varphi-\psi\|^{(h)},$$

where the function $\alpha(t)$ is continuous non-decreasing and possesses a continuous derivative for $t \geqq 0$ and where $K(t)$ is a bounded function. If for some q, $0 < q < 1$, there exists a number $T > 0$ such that, for all $t \geqq 0$,

(5) $$K(t)e^{-q[\alpha(t+T)-\alpha(t)]} \leqq 1,$$

then there exists a function $V(t,\varphi,\psi)$ continuous in t, φ, and ψ for all $t \geqq 0$, $\varphi, \psi \in c_{H_0}$, $H_0 > 0$, such that

(6) $$\|\varphi-\psi\| \leqq V(t,\varphi,\psi) \leqq K(t)V(t,\varphi,\psi).$$

(7) $$\dot{V}_{(1),(2)}(t,\varphi,\psi) \leqq -(1-q)\alpha^1(t)V(t,\varphi,\psi)$$

(8) $$|V(t,\varphi_1,\psi_1) - V(t,\varphi_2,\psi_2)| \leqq e^{LT} \sup_{0\leqq\tau\leqq T} e^{\alpha(t+\tau)-\alpha(t)}[\|\varphi_1-\varphi_2\|^{(h)} + \|\psi_1-\psi_2\|^{(h)}].$$

Proof of lemma 1. Let q, T be as defined above.

Let

(9) $$V(t,\varphi,\psi) = \sup_{\tau\geqq 0} \|x_{t+\tau}(t,\varphi) - y_{t+\tau}(t,\psi)\|^{(h)} e^{(1-q)[\alpha(t+\tau)-\alpha(t)]}$$

with the assumptions on $K(t)$ and $\alpha(t)$. $V(t,\varphi,\psi)$ is defined for all $\varphi,\psi \in c_{H_0}$, $H_0 = \frac{K}{M}$, $M = \sup_{t\geqq 0} K(t)$.

Relation (6) can easily be proved. We may prove (7) in the following manner.

$$\begin{aligned}
&\dot{V}_{(1),(2)}(t,\varphi,\psi) \\
&= \overline{\lim_{h\to 0+}} \frac{1}{h}[V(t+h, x_{t+h}(t,\varphi), y_{t+h}(t,\psi)) - V(t, x_t(t,\varphi), y_t(t,\psi))] \\
&= \overline{\lim_{h\to 0+}} \frac{1}{h}[\sup_{\tau\geqq 0} \|x_{t+h+\tau}(t+h, x_{t+h}(t,\varphi)) - y_{t+h+\tau}(t+h, y_{t+h}(t,\psi))\|^{(h)} \\
&\qquad \times e^{(1-q)[\alpha(t+h+\tau)-\alpha(t+h)]} - \sup_{\tau\geqq 0} \|x_{t+\tau}(t,\varphi) - y_{t+\tau}(t,\psi)\|^{(h)} \\
&\qquad \times e^{(1-q)[\alpha(t+\tau)-\alpha(t)]}] \\
&= \overline{\lim_{h\to 0+}} \frac{1}{h} \sup_{\tau\geqq h} \|x_{t+\tau}(t,\varphi) - y_{t+\tau}(t,\psi)\| e^{(1-q)[\alpha(t+\tau)-\alpha(t)]} \\
&\qquad - \sup_{\tau\geqq 0} \|x_{t+\tau}(t,\varphi) - y_{t+\tau}(t,\psi)\| e^{(1-q)[\alpha(t+\tau)-\alpha(t)]} \\
&\leqq \overline{\lim_{h\to 0+}} \frac{1}{h} \sup_{\tau\geqq 0} \|x_{t+\tau}(t,\varphi) - y_{t+\tau}(t,\psi)\| e^{(1-q)[\alpha(t+\tau)-\alpha(t)]} \{e^{(1-q)[\alpha(t)-\alpha(t+h)]} - 1\} \\
&= -(1-q)\alpha^1(t) \cdot V(t,\varphi,\psi).
\end{aligned}$$

Hence relation (7) is proved. Now we prove (8). We use the following lemma (lemma 1.3 from [2]) to prove (8).

Lemma A. *Suppose* $t_0 \geqq 0$, $\varphi, \varphi^* \in c_{H_1}$, $H_1 > 0$, $H_1 < H$ *are given. Then so long as the solutions* $x_t(t_0, \varphi)$ *and* $x_t(t_0, \varphi^*)$ *of* (1) $\in c_H$,

$$||x_t(t_0, \varphi) - x_t(t_0, \varphi^*)||^{(h)} \leqq e^{L(t-t_0)}||\varphi_{t_0} - \varphi_{t_0}^*||^{(h)},$$

where L is the Lipschitz constant defined above ; i.e. the solutions depend continuously upon the initial values.

Proof of lemma A is based on Krasovskii's work [3].

Proof of (8) is as follows. Using the assumption (4), we have

$$||x_{t+\tau}(t_0, \varphi) - y_{t+\tau}(t_0, \psi)||^{(h)} e^{(1-q)[\alpha(t+\tau)-\alpha(t)]}$$
$$\leqq K(t) e^{-q[\alpha(t+\tau)-\alpha(t)]}||\varphi - \psi||^{(h)}.$$

From (5) and (9) we have

$$V(t, \varphi, \psi) = \sup_{0 \leqq \tau \leqq T} ||x_{t+\tau}(t, \varphi) - y_{t+\tau}(t, \psi)||^{(h)} e^{(1-q)[\alpha(t+\tau)-\alpha(t)]}.$$

Using lemma A, we get

$$|V(t, \varphi_1, \psi_1) - V(t, \varphi_2, \psi_2)|$$
$$\leqq \sup_{0 \leqq \tau \leqq T} [||x_{t+\tau}(t, \varphi_1) - x_{t+\tau}(t, \varphi_2)||^{(h)} + ||y_{t+\tau}(t, \psi_1) - y_{t+\tau}(t, \psi_2)||^{(h)}]$$
$$\times e^{(1-q)[\alpha(t+\tau)-\alpha(t)]}$$
$$\leqq e^{LT} \cdot \sup_{0 \leqq \tau \leqq T} e^{(1-q)[\alpha(t+\tau)-\alpha(t)]}[||\varphi_1 - \varphi_2||^{(h)} + ||\psi_1 - \psi_2||^{(h)}].$$

Before we state our main theorem, we give the following definitions.

Definition 1. System (1) is said to be *stable with respect to* (2), if for every $\varepsilon > 0$ and $t_0 \geqq 0$, there exists $\delta(t_0, \varepsilon) > 0$, that is continuous in t_0 for each ε and such that

$$||\varphi - \psi|| < \delta(t_0, \varepsilon)$$

implies

$$||x_t(t_0, \varphi) - y_t(t_0, \psi)|| < \varepsilon \quad \text{for all} \quad t \geqq t_0.$$

Definition 2. The trivial solution of (1) is said to be *L^p-stable with respect to the trivial solution of* (2), if the definition 1 holds and if for all $t_0 \geqq 0$, there exists a $\delta_0 = \delta_0(t_0) > 0$ such that

$$||\varphi - \psi|| < \delta_0$$

implies

$$\int_{t_0}^{\infty} ||x_t(t, \varphi) - y_t(t_0, \psi)||^p dt < \infty$$

for all $t \geqq t_0$, where $x_t(t_0, \varphi)$ and $y_t(t_0, \psi)$ are the solutions of (1) and (2) respectively with their initial values φ and ψ respectively, at time t_0.

Theorem. *If* $V(t, x_t(t_0, \varphi), y_t(t_0, \psi))$ *satisfies the above assumptions given in lemma* 1. *Let V be such that its derivative along the solutions of* (1) *and* (2) *is*

$$\dot{V}_{(1),(2)}(t, x_t(t_0, \varphi), y_t(t_0, \psi)) \leqq -c||x_t(t_0, \varphi) - y_t(t_0, \psi)||^p$$

for all $t \geqq t_0$, $\varphi, \psi \in c_{H_0}$ $H_0 > 0$ *and for some* $c > 0$, $p > 0$. *Then the zero solu-*

tion of (1) *is* L^p*-stable with respect to the zero solution of* (2).

Proof of theorem. With assumptions of $V(t, x_t(t_0, \varphi), y_t(t_0, \psi))$, the zero solution of (1) is stable with respect to the zero solution of (2), follows from the work of Hale [2].

Now let

$$\gamma(t) = V(t, x_t(t_0, \varphi), y_t(t_0, \psi)) + c\int_{t_0}^{t} ||x_t(t_0, \varphi) - y_t(t_0, \psi)||^p dt$$

for all $t \geqq t_0$. Let t be fixed in $[t_0, \infty)$,

$$\overline{\lim_{h\to 0+}} \frac{1}{h}[\gamma(t+h) - \gamma(t)]$$

$$\leqq \overline{\lim_{h\to 0+}} \frac{1}{h}[V(t+h, x_{t+h}(t, \varphi), y_{t+h}(t, \psi)) - V(t, x_t(t_0, \varphi), y_t(t_0, \psi))]$$

$$+ c||x_t(t_0, \varphi) - y_t(t_0, \psi)||^p$$

$$= \dot{V}_{(1),(2)}(t, x_t(t_0, \varphi), y_t(t_0, \psi)) + c\cdot||x_t(t_0, \varphi) - y_t(t_0, \psi)||^p \leqq 0.$$

Hence $\gamma(t)$ is non-increasing in $[t_0, \infty)$. But $\gamma(t_0) = V(t_0, \varphi, \psi)$.

Therefore $\gamma(t) \leqq V(t_0, \varphi, \psi)$ for all $t \geqq t_0$. Hence

$$0 \leqq V(t, x_t(t_0, \varphi), y_t(t_0, \psi))$$

$$\leqq -c\int_{t_0}^{t} ||x_t(t_0, \varphi) - y_t(t_0, \psi)||^p dt + V(t_0, \varphi, \psi)$$

for all $t \geqq t_0$, so that

$$\int_{t_0}^{\infty} ||x_t(t_0, \varphi) - y_t(t_0, \psi)||^p dt \leqq \frac{1}{c}\cdot V(t_0, \varphi, \psi)$$

Hence the theorem follows.

References

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