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Note on Perturbation and Degeneration of Abstract Differential Equations in Banach Space

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Perturbation and degeneration of abstract differential equations in a Banach space was treated systematically by Prof. M. Nagumo in [1], where a necessary and sufficient condition in order that a solution of

$(1)_0$ $$du(t)/dt+A_0(t)u(t)=f_0(t)$$

(resp. $(2)_0$ $A_0(t)u(t)=f_0(t)$)

is completely stable with respect to the equation

$(1)_\varepsilon$ $$du(t)/dt+A_\varepsilon(t)u(t)=f_\varepsilon(t)$$

(resp. $(2)_\varepsilon$ $\varepsilon\, du(t)/dt+A_\varepsilon(t)u(t)=f_\varepsilon(t)$)

for $\varepsilon\rightarrow 0$ was given in a general form. Roughly speaking the complete stability means that the solution $u_\varepsilon(t)$ of $(1)_\varepsilon$ (resp. $(2)_\varepsilon$) converges uniformly in $s\leqq t$ to that $u_0(t)$ of $(1)_0$ (resp. $(2)_0$) as $\varepsilon\rightarrow 0$ when the initial value of u_ε at $t=s$ tends to that of u_0 at $t=s$ and $f_\varepsilon(t)$ converges uniformly to $f_0(t)$; the uniformity in the convergence of u_ε is required also with respect to the initial time s. Usually $A_0(t)$ is weaker than $A_\varepsilon(t)$, $\varepsilon>0$, namely $D(A_0(t))\supset D(A_\varepsilon(t))$ if $\varepsilon>0$.

The object of the present paper is to give an example of a complete stable perturbation and degeneration. The example, which is rather artificial, is the same one that was investigated in [2]. However, only the weak convergence of the solution of $(1)_\varepsilon$ was investigated there, and the result will be completed in this paper by showing that the same thing remains valid with regard to strong convergence. The equations in the example are integrable and the solutions can be expressed making use of the fundamental solution which can be constructed by the method of [2], and hence the greater part of the proof is occupied by examining the convergence of the fundamental solution of $(1)_\varepsilon$ (resp. $(2)_\varepsilon$) as $\varepsilon\rightarrow 0$.

1. Preliminaries. Throughout this paper we denote by X a Banach space. We use only the strong topology of X unless otherwise stated and denote "convergence in the strong (operator) topology", "uniform convergence in the strong (operator) topology" by the symbols "$\rightarrow$", "$\Rightarrow$" respectively. For an operator T of X into itself $D(T)$ and $R(T)$ denote the domain and the range of T respectively. The following theorem is proved in [2].

Theorem A. *Let $A(t)$, $0\leqq t\leqq T$, be densely defined closed linear operators in X. Suppose that $\{A(t)\}$ satisfies the following assumptions:*
(I) *for each $t\in[0, T]$ the resolvent set of $A(t)$ contains a fixed closed angular domain $\sum=\{\lambda: \arg\lambda\notin(-\theta_0, \theta_0)\}$ where θ_0 is a positive number with $0<\theta_0<\pi/2$. For any $t\in[0, T]$ and $\lambda\in\sum$, $\|(\lambda-A(t))^{-1}\|\leqq M/|\lambda|$ where M is a constant independent of λ and t;*
(II) *$A(t)^{-1}$ is continuously differentiable in t in the uniform operator topology;*
(III) *there exists a positive number $\rho\leqq 1$ such that $R(dA(t)^{-1}/dt)\subset D(A(t)^{\rho})$ and $A(t)^{\rho}\cdot dA(t)^{-1}/dt$ is strongly continuous in $0\leqq t\leqq T$. Hence with some positive constant N independent of t we have $\|A(t)^{\rho}\cdot dA(t)^{-1}/dt\|\leqq N$.*
Then there exists a fundamental solution $U(t, s)$, $0\leqq s\leqq t\leqq T$, of the equation

$$du(t)/dt+A(t)u(t)=f(t): \tag{1.1}$$

if $s<t$, $R(U(t, s))\subset D(A(t))$ and $U(t, s)$ satisfies

$$\partial U(t, s)/\partial t+A(t)U(t, s)=0, \quad 0\leqq s<t\leqq T, \quad U(s, s)=I,$$

$$\left\|\frac{\partial}{\partial t}U(t, s)\right\|=\|A(t)U(t, s)\|\leqq\frac{C}{t-s}.$$

If $f(t)$ is strongly Hölder continuous, then the unique solution of (1.1) in $s<t\leqq T$ satisfying $u(s)=u$ is given by

$$u(t)=U(t, s)u+\int_s^t U(t, \sigma)f(\sigma)d\sigma. \tag{1.2}$$

By (I) $A(t)$ generates an analytic semi-group

$$\exp(-\sigma A(t))=\frac{-1}{2\pi i}\int_{\Gamma}e^{-\lambda\sigma}(\lambda-A(t))^{-1}d\lambda, \quad \sigma>0, \tag{1.3}$$

where Γ is a smooth path running in $\sum$ from $\infty e^{-\theta_0 i}$ to $\infty e^{\theta_0 i}$. In what follows we use C to denote constants depending only on T, θ_0, M, ρ, N which appeared in the assumptions (I), (II), (III) of Theorem A. The fundamental solution $U(t, s)$ of (1.1) is constructed in the following manner:

$$U(t, s)=\exp(-(t-s)A(t))+W(t, s), \tag{1.4}$$

$$W(t, s)=\int_s^t\exp(-(t-\tau)A(t))R(\tau, s)d\tau, \tag{1.5}$$

$$R(t, s)=\textstyle\sum_{m=0}^{\infty}R_m(t, s), \tag{1.6}$$

$$\begin{aligned}R_1(t, s)&=(\partial/\partial t+\partial/\partial s)\exp(-(t-s)A(t))\\&=\frac{-1}{2\pi i}\int_{\Gamma}e^{-\lambda(t-s)}\frac{\partial}{\partial t}(\lambda-A(t))^{-1}d\lambda,\end{aligned} \tag{1.7}$$

$$R_m(t, s)=\int_s^t R_1(t, \tau)R_{m-1}(\tau, s)d\tau, \quad \text{for} \quad m=2, 3, \cdots. \tag{1.8}$$

As is easily seen for each m

$$\|R_m(t, s)\|\leqq C^m\Gamma(\rho)^m(t-s)^{m\rho-1}/\Gamma(m\rho).$$

In what follows $\{A_{\varepsilon}(t); 0\leqq t\leqq T, 0\leqq\varepsilon\leqq\varepsilon_0\}$ is to be a family of closed linear operators in X which satisfy the asumptions (I), (II), (III) of Theorem A with constants θ_0, M, ρ, N independent of t and ε.

2. Singular perturbation for $(1)_\varepsilon$. In this section we investigate the behaviour of the solution of $(1)_\varepsilon$ as $\varepsilon \downarrow 0$.

Theorem 1. *Let* $A_0(t)$, $0 \leqq t \leqq T$, *be a family of closed linear operators in* X *which satisfy the assumptions* (I), (II) *of Theorem* A. *Suppose that* $D(A_0(t)^{\rho'}) \supset R(dA_0(t)^{-1}/dt)$ *for some positive number* $\rho' \leqq 1$. *We assume also that*

(a) $$\exp(-\delta A_0(t)) \Rightarrow I \quad in \quad 0 \leqq t \leqq T \quad as \quad \delta \downarrow 0,$$

(b) $$A_\varepsilon(t)^{-1} \Rightarrow A_0(t)^{-1} \quad in \quad 0 \leqq t \leqq T \quad as \quad \varepsilon \downarrow 0,$$

(c) $$dA_\varepsilon(t)^{-1}/dt \Rightarrow dA_0(t)^{-1}/dt \quad in \quad 0 \leqq t \leqq T \quad as \quad \varepsilon \downarrow 0.$$

Then for each $s \in [0, T)$, $u_0 \in X$ *and a Hölder continuous function* $f_0(t)$, $0 \leqq t \leqq T$, *with values in* X *a solution* $u_0(t\,;s)$ *of the initial value problem*

$$du(t)/dt + A_0(t)u(t) = f_0(t), \qquad s < t \leqq T, \qquad u(s) = u_0,$$

exists in $s \leqq t \leqq T$ *and is unique there. Furthermore, the solution* $u_\varepsilon(t\,;s)$ *of the initial value problem*

$$du(t)/dt + A_\varepsilon(t)u(t) = f_\varepsilon(t), \qquad s < t \leqq T, \qquad u(s) = u_\varepsilon,$$

converges to $u_0(t\,;s)$ *in the following manner:*

$$u_\varepsilon(t\,;s) \Rightarrow u_0(t\,;s) \quad in \quad 0 \leqq s \leqq t \leqq T,$$

$$\left.\begin{array}{l} du_\varepsilon(t\,;s)/dt \Rightarrow du_0(t\,;s)/ds \\ A_\varepsilon(t)u_\varepsilon(t\,;s) \Rightarrow A_0(t)u_0(t\,;s) \end{array}\right\} \quad in \quad \delta \leqq s+\delta \leqq t \leqq T,$$

for any $\delta > 0$, *provided that*

(i) $u_\varepsilon \to u_0$ *as* $\varepsilon \downarrow 0$,

(ii) $f_\varepsilon(t) \Rightarrow f_0(t)$ *in* $0 \leqq t \leqq T$ *as* $\varepsilon \downarrow 0$,

(iii) $$\left.\begin{array}{l} \|f_\varepsilon(t) - f_\varepsilon(r)\| \leqq K(t-r)^\alpha \\ \|f_0(t) - f_0(r)\| \leqq K(t-r)^\alpha \end{array}\right\} \quad in \quad 0 \leqq r \leqq t \leqq T,$$

where K *and* α *are positive constants independent of* t, r *and* ε.

Remark. The assumption (a) is satisfied if for any g in some dense subset of X there exists a sequence $\{g_n(t)\} \subset D(A_0(t))$ such that $A_0(t)g_n(t)$ is continuous in $0 \leqq t \leqq T$ and $g_n(t) \Rightarrow g$ in $0 \leqq t \leqq T$ as $n \to \infty$.

Proof of Theorem 1. Let $U^\varepsilon(t, s)$, $W^\varepsilon(t, s)$, $R^\varepsilon(t, s)$, $R^\varepsilon_m(t, s)$, $m = 1, 2, \cdots$, be operator valued functions with $A_\varepsilon(t)$ in place of $A(t)$ in the definition of the corresponding functions in (1.4)–(1.8). In the proof of the present theorem we often use the following elementary lemma whose proof might be omitted.

Lemma 1. *Let* Λ *be some set and let* $T_\varepsilon(t)$, $\varepsilon > 0$, $t \in \Lambda$, *be a family of bounded operators in* X *which converges to* 0 *as* $\varepsilon \downarrow 0$ *in the strong operator topo-*

logy uniformly in Λ. Then for any compact subset K of X $\{T_\varepsilon(t)u\}$ converges to 0 in the strong operator topology as $\varepsilon\downarrow 0$ uniformly in $t\in\Lambda$ and $u\in K$.

Lemma 2. *Under the assumptions of Theorem 1*

$$\exp(-(t-s)A_\varepsilon(t))\Rightarrow\exp(-(t-s)A_0(t))$$

in $0\leqq s\leqq t\leqq T$.

Proof. Let g be an arbitrary element of X which will be fixed throughout the proof. Let C_1 be a constant such that

$$\|\exp(-(t-s)A_\varepsilon(t))\|\leqq C_1$$

for $0\leqq s\leqq t\leqq T$ and $\varepsilon\geqq 0$. For any $\delta>0$

$$\begin{aligned}(2.1)\qquad &\exp(-(t-s)A_\varepsilon(t))g-\exp(-(t-s)A_0(t))g\\ &=\exp(-(t-s)A_\varepsilon(t))\{I-\exp(-\delta A_0(t))\}g\\ &+\exp(-(t-s)A_\varepsilon(t))\{\exp(-\delta A_0(t))-\exp(-\delta A(t))\}g\\ &+\{\exp(-(t-s+\delta)A_\varepsilon(t))-\exp(-(t-s+\delta)A_0(t))\}g\\ &+\exp(-(t-s)A_0(t))\{\exp(-\delta A_0(t))-I\}g.\end{aligned}$$

For any given $\eta>0$ there exist by assumption a positive number δ depending only on η such that for any $t\in[0, T]$

$$(2.2)\qquad C_1\|\{I-\exp(-\delta A_0(t))\}g\|<\eta/4.$$

For any such δ the first and the last term of (2.1) are dominated by $\eta/4$ in norm for any $t\in[0, T]$. Using (1.3) and noting the uniform boundedness of $\|A_\varepsilon(t)(\lambda-A_\varepsilon(t))^{-1}\|$ we get

$$\begin{aligned}(2.3)\qquad &\left.\begin{aligned}&C_1\|\{\exp(-\delta A_0(t))-\exp(-\delta A_\varepsilon(t))\}g\|\\ &\|\{\exp(-(t-s+\varepsilon)A_\varepsilon(t)-\exp(-(t-s+\delta)A_0(t))\}g\|\end{aligned}\right\}\\ &\leqq C\int_{\lambda\in\Gamma,\,|\lambda|\leqq N} e^{-\delta\operatorname{Re}\lambda}\|\{A_\varepsilon(t)^{-1}-A_0(t)^{-1}\}A_0(t)(\lambda-A_0(t))^{-1}g\|\,|d\lambda|\\ &+C\int_{\lambda\in\Gamma,\,|\lambda|\geqq N} e^{-\delta\operatorname{Re}\lambda}\{\|(\lambda-A_\varepsilon(t))^{-1}g\|+\|(\lambda-A_0(t))^{-1}g\|\}\,|d\lambda|\\ &\leqq CN\sup_{\lambda\in\Gamma,\,|\lambda|\leqq N}\|\{A_\varepsilon(t)^{-1}-A_0(t)^{-1}\}A_0(t)(\lambda-A_0(t))^{-1}g\|+\frac{C\|g\|}{N\delta}.\end{aligned}$$

Let N be so large that $8C\|g\|<N\delta$. Clearly such N can be chosen depending only on η. Then the subset

$$K=\{A_0(t)(\lambda-A_0(t))^{-1}g: 0\leqq t\leqq T, \lambda\in\Gamma, |\lambda|\leqq N\}$$

of X is a compact subset determined only by η. Then it follows from Lemma 1 that there exists a positive constant $\varepsilon(\eta)$ depending only on η such that for any $t\in[0, T]$ and $h\in K$

$$\|\{A_\varepsilon(t)^{-1}-A_0(t)^{-1}\}h\|<\eta/8CN.$$

Hence the first term on the right of (2.3) is dominated by $\eta/8$. Thus we get

$$\|\exp(-(t-s)A_\varepsilon(t))g-\exp(-(t-s)A_0(t))g\|<\eta$$

which completes the proof of Lemma 2.

In order to show that $A_0(t)$, $0 \leqq t \leqq T$, satisfies the assumption (III) of Theorem A it is sufficient to prove the following lemma in which we write ρ instead of $\min(\rho, \rho')$.

Lemma 3. *For any* ρ_1 *with* $0 < \rho_1 < \rho$

$$A_\varepsilon(t)^{\rho_1} \cdot dA_\varepsilon(t)^{-1}/dt \Rightarrow A_0(t)^{\rho_1} \cdot dA_0(t)^{-1}/dt$$

in $0 \leqq t \leqq T$ *as* $\varepsilon \downarrow 0$.

Proof. The lemma is easily proved using that

$$A_\varepsilon(t)^{\rho_1} \cdot dA_\varepsilon(t)^{-1}/dt = A_\varepsilon(t)^{\rho_1 - \rho} A_\varepsilon(t)^{\rho} \cdot dA_\varepsilon(t)^{-1}/dt$$

$$= \frac{\sin \pi(\rho - \rho_1)}{\pi} \int_0^\infty \lambda^{\rho_1 - \rho} (\lambda + A_\varepsilon(t))^{-1} A_\varepsilon(t)^{\rho} \cdot dA_\varepsilon(t)^{-1}/dt \, d\lambda$$

and the commutativity of $(\lambda + A_\varepsilon(t))^{-1}$ and $A_\varepsilon(t)^{\rho}$, $\varepsilon \geqq 0$.

Lemma 4. *For any fixed* $\delta > 0$ $R^\varepsilon(t, s) \Rightarrow R^0(t, s)$ *as* $\varepsilon \downarrow 0$ *in* $\delta \leqq s + \delta \leqq t \leqq T$.

Proof. Let g be an arbitrary element of X which will be fixed throughout the Proof. Let us show by induction that for any m and $\delta > 0$

$$R_m^\varepsilon(t, s) \Rightarrow R_m^0(t, s) \tag{2.4}$$

in $\delta \leqq s + \delta \leqq t \leqq T$. That (2.4) is true for $m = 1$ can be proved just as Lemma 2. Suppose that (2.4) is true for some $m > 1$. When $t - s \geqq \delta$ and $0 < \delta' < \delta$

$$\begin{aligned} & \| R_{m+1}^\varepsilon(t, s) g - R_{m+1}^0(t, s) g \| \\ & \leqq \frac{2C^{m+1} \Gamma(\rho_1)^m}{\Gamma(m \rho_1)} \Big(\int_s^{s+s'} + \int_{t-s'}^t \Big) (t - \sigma)^{\rho_1 - 1} (\sigma - s)^{m \rho_1 - 1} d\sigma \, \| g \| \\ & + C \sup_{\sigma - s \geqq \delta'} \| \{ R_m^\varepsilon(\sigma, s) - R_m^0(\sigma, s) \} g \| \\ & + C \sup_{s + \delta' \leqq \sigma \leqq t - \delta'} \| \{ R_1^\varepsilon(t, \sigma) - R_1^0(t, \sigma) \} R_m^0(\sigma, s) g \|. \end{aligned} \tag{2.5}$$

Let η be any given positive number. Then it is easy to show that if $\delta' = \delta'(\eta, \delta)$ is sufficiently small depending only on η and δ (except m) the first term on the right of (2.5) is dominated by $\eta/3$. By the induction hypothesis if ε is sufficiently small depending only on η and $\delta' = \delta'(\eta, \delta)$ the second term on the right of (2.5) is also dominated by $\eta/3$. Noting that $\{ R_m^0(\sigma, s) g : \delta' \leqq s + \delta' \leqq \sigma \leqq T \}$ is a compact subset depending only on $\delta' = \delta'(\eta, \delta)$ we get with the aid of Lemma 1 that the last term on the right of (2.5) is dominated by $\eta/3$ provided that ε is sufficiently small depending only on η and $\delta' = \delta'(\eta, \delta)$. q.e.d.

With the aid of Lemmas 2, 3, 4 we get without difficulty

Lemma 5. *As* $\varepsilon \downarrow 0$

$$W^\varepsilon(t, s) \Rightarrow W^0(t, s), \quad U^\varepsilon(t, s) \Rightarrow U^0(t, s)$$

in $0 \leqq s \leqq t \leqq T$.

End of the proof of Theorem 1. That $u_\varepsilon(t, s) \Rightarrow u_0(t, s)$ is easily verified by (1.2) and Lemma 5. The remaining part of the theorem can be proved without difficulty, and the proof may be omitted.

3. Degeneration of $(2)_\varepsilon$. In this section we investigate the behaviour of the solution of $(2)_\varepsilon$ as $\varepsilon\downarrow 0$.

Theorem 2. *Suppose that* $A_0(t)^{-1}$ *exists and is strongly continuous in* $0\leqq t\leqq T$. *Suppose also that* $A_\varepsilon(t)^{-1}\Rightarrow A_0(t)^{-1}$ *in* $0\leqq t\leqq T$ *as* $\varepsilon\downarrow 0$. *Then the solution* $u_\varepsilon(t\,;s)$ *of*

$$\varepsilon\, du/dt+A_\varepsilon(t)u(t)=f_\varepsilon(t), \qquad u(s)=u_\varepsilon(s)$$

converges to the solution $u_0(t)=A_0(t)^{-1}f_0(t)$ *of*

$$A_0(t)u(t)=f_0(t)$$

in the following manner:

(3.1) $$u_\varepsilon(t\,;s)\Rightarrow u_0(t) \quad in \quad 0\leqq s\leqq t\leqq T,$$

(3.2) $$A_\varepsilon(t)u_\varepsilon(t\,;s)\Rightarrow A_0(t)u(t) \quad in \quad \delta\leqq s+\delta\leqq t\leqq T$$

for any $\delta>0$ *provided that*

(i) $u_\varepsilon(s)\Rightarrow A_0(s)^{-1}f_0(s)$ *in* $0\leqq s\leqq T$ *as* $\varepsilon\downarrow 0$,

(ii) $f_\varepsilon(t)\Rightarrow f_0(t)$ *in* $0\leqq t\leqq T$ *as* $\varepsilon\downarrow 0$,

(iii) $\|f_\varepsilon(t)-f_\varepsilon(r)\|\leqq K(t-r)^\alpha$ *for* $0\leqq r\leqq t\leqq T$,

where the constants K *and* α *are independent of* t, r *and* ε.

Proof. As is easily seen $\tilde{A}_\varepsilon(t)=\varepsilon^{-1}A_\varepsilon(t)$ satisfies the assumptions (I), (II), (III) of Theorem A with θ_0, M, ρ unchanged and with N replaced by $\varepsilon^{1-\rho}N$. Therefore the fundamental solution

$$\tilde{U}_\varepsilon(t,s)=\exp(-(t-s)\tilde{A}_\varepsilon(t))+\tilde{W}_\varepsilon(t,s)$$

of

$$du(t)/dt+\tilde{A}_\varepsilon(t)u(t)=0$$

is constructed as in (1.4)-(1.8). Following [2; pp. 242-243] we get

$$\|\tilde{W}_\varepsilon(t,s)\|\leqq C\varepsilon^{1-\rho}(t,s)^\rho,$$

$$\|\tilde{A}_\varepsilon(t)\tilde{W}_\varepsilon(t,s)\|\leqq C\varepsilon^{1-\rho}(t-s)^{\rho-1}.$$

The solution $u_\varepsilon(t\,;s)$ is given by

$$u_\varepsilon(t\,;s)=\tilde{U}_\varepsilon(t,s)u_\varepsilon(s)+\varepsilon^{-1}\int_s^t\tilde{U}_\varepsilon(t,\sigma)f_\varepsilon(\sigma)d\sigma.$$

Noting

$$\left\|\varepsilon^{-1}\int_s^t\tilde{W}_\varepsilon(t,\sigma)f_\varepsilon(\sigma)d\sigma\right\|=\left\|A_\varepsilon(t)^{-1}\int_s^t\tilde{A}_\varepsilon(t)\tilde{W}_\varepsilon(t,\sigma)f_\varepsilon(\sigma)d\sigma\right\|$$

$$\leqq C\int_s^t\varepsilon^{1-\rho}(t-\sigma)^{\rho-1}\|f_\varepsilon(\sigma)\|\,d\sigma$$

we get

$$\tilde{W}_\varepsilon(t,s)u_\varepsilon(s)+\varepsilon^{-1}\int_s^t\tilde{W}_\varepsilon(t,\sigma)f_\varepsilon(\sigma)d\sigma\Rightarrow 0$$

in $0\leqq s\leqq t\leqq T$. To conclude (3.1) it remains to show that

(3.3) $$\left.\begin{aligned}&\exp(-(t-s)\tilde{A}_\varepsilon(t))u_\varepsilon(s)+\varepsilon^{-1}\int_s^t\exp(-t-\sigma)\tilde{A}_\varepsilon(t))f_\varepsilon(\sigma)d\sigma\\&\qquad\Rightarrow A_0(t)^{-1}f_0(t)\quad\text{in}\quad 0\leqq s\leqq t\leqq T.\end{aligned}\right\}$$

Noting

$$\varepsilon^{-1}\int_s^t \exp(-(t-\sigma)\tilde{A}_\varepsilon(t))d\sigma = A_\varepsilon(t)^{-1}\{I-\exp(-(t-s)\tilde{A}_\varepsilon(t))\}$$

we get

$$\exp(-(t-s)\tilde{A}_\varepsilon(t))u_\varepsilon(s)+\varepsilon^{-1}\int_s^t \exp(-(t-\sigma)\tilde{A}_\varepsilon(t))f_\varepsilon(\sigma)d\sigma$$

$$(3.4)\quad = A_\varepsilon(t)^{-1}f_\varepsilon(t)+\exp(-(t-s)\tilde{A}_\varepsilon(t))\{u_\varepsilon(s)-A_\varepsilon(t)^{-1}f_\varepsilon(t)\}$$
$$+\varepsilon^{-1}\int_s^t \exp(-(t-\sigma)\tilde{A}_\varepsilon(t))(f_\varepsilon(\sigma)-f_\varepsilon(t))d\sigma.$$

The norm of the last term on the right of (3.4) is dominated by

$$\left\|\varepsilon^\gamma A_\varepsilon(t)^{-\gamma-1}\int_s^t \tilde{A}_\varepsilon(t)^{\gamma+1}\exp(-(t-\sigma)\tilde{A}_\varepsilon(t))(f_\varepsilon(\sigma)-f_\varepsilon(t))d\sigma\right\|$$
$$\leqq CK\varepsilon^\gamma\int_s^t (t-\sigma)^{\alpha-\gamma-1}d\sigma \leqq CK\varepsilon^\gamma(\alpha-\gamma)^{-1}(t-s)^{\alpha-\gamma}.$$

for any γ with $0<\gamma<\alpha$. As for the second term

$$\|\exp(-(t-s)\tilde{A}_\varepsilon(t))\{u_\varepsilon(s)-A_\varepsilon(t)^{-1}f_\varepsilon(t)\}\|$$
$$\leqq C\|u_\varepsilon(s)-A_0(s)^{-1}f_0(s)\|+C\|A_\varepsilon(t)^{-1}f_\varepsilon(t)-A_0(t)^{-1}f_0(t)\|$$
$$+\|\exp(-(t-s)\tilde{A}_\varepsilon(t))\{A_0(t)^{-1}f_0(t)-A_0(s)^{-1}f_0(s)\}\|.$$

The proof of (3.3) is completed by treating the last term of the above inequality with the aid of the following elementary lemma.

Lemma 6. *If $F(t)$ is a strongly continuous function in $0\leqq t\leqq T$ with values in X, then*

$$\exp(-(t-s)\tilde{A}_\varepsilon(t))(F(t)-F(s))\Rightarrow 0$$

in $0\leqq s\leqq t\leqq T$ as $\varepsilon\downarrow 0$.

It is straightforward to show (3.2), and hence the proof may be omitted.

4. **Example.** In this section an example to which Theorems 1 and 2 can be applied is given. It is not trivial but should be admitted to be artificial. The example for $(1)_\varepsilon$ is the same one that was treated in [2] and is the following initial-boundary value problem

$$\frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 u}{\partial x^2}+\frac{\varepsilon}{x-t}\frac{\partial u}{\partial x}+\frac{u}{(x-t)^2}=f,\qquad a<x<b,$$
$$u(t,a)=u(t,b)=0$$

where it is assumed that $-\infty<a<0<T<b<\infty$. We choose $X=L^2[a,b]$ as the basic Banach space. For each t set

$$V(t)=\left\{u\in L^2[a,b]:\frac{du}{dx},\frac{u}{x-t}\in L^2[a,b], u(a)=u(b)=0\right\}$$

where the derivatives in the above are taken in the distribution sence. The operator $A_\varepsilon(t)$ is defined in the following manner: $u\in V(t)$ belongs to $D(A_\varepsilon(t))$

and $A_\varepsilon(t)u=f\in L^2[a,b]$ if for any $v\in V(s)$

$$\int_a^b\left\{\varepsilon\frac{du}{dx}\frac{d\bar{v}}{dx}+\varepsilon\frac{du}{dx}\frac{\bar{v}}{x-t}+\frac{u\bar{v}}{(x-t)^2}\right\}dx=\int_a^b f\bar{v}\,dx.$$

It was shown in [2] that $A_\varepsilon(t)$, $0\leqq t\leqq T$, $0<\varepsilon\leqq 4/5$, satisfies the assumptions (I), (II), (III) uniformly in t and ε with $\rho=1/2$. It is also possible to express the solution of $A_\varepsilon(t)u=g$ explicity and the formula is given in [2]. Let $A_0(t)$ be the operator

$$(A_0(t)u)(x)=u(x)/(x-t)^2.$$

If $g\in C_0^\infty(a,b)$ we can integrate by part in all the integrals of the formula for $A_\varepsilon(t)^{-1}g$ and easily show that

$$A_\varepsilon(t)^{-1}g\Rightarrow A_0(t)^{-1}g,\qquad dA_\varepsilon(t)^{-1}g/dt\Rightarrow dA_0(t)^{-1}g/dt.$$

Since $C_0^\infty(a,b)$ is dense in $L^2[a,b]$ the same things remain valid for any $g\in L^2[a,b]$. To verify that the assumption (a) of Theorem 1 is satisfied it suffices to show that a sequence $\{g_n(t)\}$ as was mentioned in the remark just after the theorem can be constructed to any $g\in L^2[a,b]$. It is elementary to see that the sequence defined by $g_n(t)=(I+n^{-1}A_0(t))^{-1}g$ has such a property. Thus all the assumptions of Theorems 1 and 2 are satisfied by $\{A_\varepsilon(t)\}$ of the present example.

References

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[2] H. Tanabe, Note on singular perturbation for abstract differential equations, Osaka J. Math., **1** (1964), 239–252.

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