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# On the Boundedness of Solutions of Ordinary Differential Equations

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1. In 1964 W.A. Coppel [1] made an interesting application of a well-known theorem of J.L. Massera and J.J. Schäffer ([4], Theorem 4.2) in order to give a necessary and sufficient condition for the existence of at least one bounded solution of the ordinary differential equation

$$\dot{x}-A(t)x=b(t) \tag{1}$$

for every $b$ in a certain class. Namely he considered the case $b\in C$, the Banach space of all bounded continuous vector functions $b$ defined on $J=[0,+\infty)$, with the norm $|b|_C=\sup_{t\in J}\|b(t)\|$, $\|\ \|$ the Euclidean norm in $R^n$.

Subsequently, in his fine book [3] (Chapter V) he considered also the case $b\in L^1$, the Banach space of all vector functions which are Lebesgue integrable on $J$, with the norm $|b|_{L^1}=\int_0^\infty\|b(t)\|\,dt$.

The case $b\in C$ was recently dealt with also by T.F. Bridgland, Jr. [1].

In what follows we want to show that the same arguments used by Coppel provide, with minor modifications, a general result (Theorem 1) for $b\in L^p$, $1\leq p\leq\infty$, the Banach space of all vector functions $b$, with $\|b(t)\|^p$ Lebesgue integrable on $J$, and the norm $|b|_{L^p}=\left(\int_0^\infty\|b(t)\|^p\,dt\right)^{1/p}$. We shall further consider, in particular, the two extreme situations arising, respectively, when all the solutions of (1) are bounded (Theorem 2), or when only one solution is bounded (Theorem 3).

2. Using the same notations as in Coppel's book, let $X_1$ be the subspace of points in $R^n$ which are values for $t=0$ of bounded solutions of

$$\dot{x}-A(t)x=0, \tag{2}$$

where $A(t)$ is an $n$ by $n$ matrix with real elements, Lebesgue integrable over finite subintervals of $J$. Let further $X_2$ be any fixed subspace of $R^n$ supplementary to $X_1$, and let $P_1$, $P_2$ denote the corresponding projections of $R^n$ onto $X_1$, $X_2$. Finally, let $Y(t)$ be the fundamental matrix of (2) for which $Y(0)=I$, the identity $n$ by $n$ matrix. We have then

**Theorem 1.** *The equation* (1) *has at least one bounded solution for every* $b\in L^p$, $1\leq p\leq\infty$, *if and only if there is a* $K>0$ *such that, for* $t\geq 0$

$$\left[\int_0^t|Y(t)P_1Y^{-1}(s)|^q\,ds+\int_t^\infty|Y(t)P_2Y^{-1}(s)|^q\,ds\right]^{1/q}\leq K, \tag{3}$$

*where, as usual,* $q=p/(p-1)$ *for* $1<p<\infty$, $q=1$ *for* $p=\infty$, $q=\infty$ *for* $p=1$, *and* $|\ |$ *is any norm of* $n$ *by* $n$ *matrices.*

**Suff.** For every $b\in L^p$

$$x(t)=\int_0^t Y(t)P_1Y^{-1}(s)b(s)ds-\int_t^\infty Y(t)P_2Y^{-1}(s)b(s)ds$$

is a solution of (1), and it is bounded by virtue of (3), since

$$\|x(t)\|\le\left(\int_0^t |Y(t)P_1Y^{-1}(s)|^q ds\right)^{1/q}|b|_{L^p}$$
$$+\left(\int_t^\infty |Y(t)P_2Y^{-1}(s)|^q ds\right)^{1/q}|b|_{L^p}\le 2K|b|_{L^p}.$$

**Nec.** We closely follow Coppel's arguments up to a certain point. Define

$$L(t,s)=\begin{cases} Y(t)P_1Y^{-1}(s) & \text{for } 0\le s\le t\\ -Y(t)P_2Y^{-1}(s) & \text{for } 0\le t\le s\end{cases}$$

and let $b\in L^p$ be a function identically zero for $t\ge T$, arbitrarily fixed and $>0$. Then

$$x(t)=\int_0^T L(t,s)b(s)ds$$

is a solution of (1), it is bounded, since

$$x(t)=Y(t)P_1\int_0^T Y^{-1}(s)b(s)ds \quad \text{for} \quad t\ge T,$$

and we have $x(0)\in X_2$, since

$$x(0)=-P_2\int_0^T Y^{-1}(s)b(s)ds.$$

Now Massera-Schäffer's theorem asserts that there exists a constant $K>0$ such that for every $b\in L^p$, equation (1) has a unique bounded solution $x(t)$ with $x(0)\in X_2$ and this solution satisfies $|x|_C\le K|b|_{L^p}$. Therefore we have

$$\left\|\int_0^T L(t,s)b(s)ds\right\|\le K|b|_{L^p},\quad 0\le t \tag{4}$$

with $b\in L^p$, $b=0$ for $t\ge T$ and $K>0$ independent of $T$.

At this point let $y'$ denote the transpose of $y\in R^n$. Then (4) can be written as

$$\left|\int_0^T y'L(t,s)\,b(s)ds\right|\le K|b|_{L^p},\quad 0\le t,\quad \|y\|\le 1.$$

Hence (Cfr. Bourbaki, Intégration, Ch. IV, p. 211)

$$\left(\int_0^T \|y'L(t,s)\|^q ds\right)^{1/q}\le K,\quad 0\le t,\quad \|y\|\le 1.$$

Therefore if $y$ ranges over the $n$ unit vectors we have, for some $K'>0$ independent of $T$

$$\left(\int_0^T \|L(t,s)\|^q ds\right)^{1/q} \leq K', \quad 0 \leq t$$

where $\|L\|$ is the Euclidean norm of $L$, hence also

$$\left(\int_0^T |L(t,s)|^q ds\right)^{1/q} \leq K'', \quad 0 \leq t$$

for some $K''>0$, still independent of $T$, and $|L|$ any norm of $L$. Finally we get

$$\left(\int_0^\infty |L(t,s)|^q ds\right)^{1/q} \leq K'', \quad 0 \leq t$$

i.e. (3) with $K$ replaced by $K''$.

3. From Th. 1 follows

**Theorem 2.** *For each $b \in L^p$, $1 \leq p \leq \infty$, all the solutions of* (1) *are bounded if and only if*

$$\left(\int_0^t |Y(t)Y^{-1}(s)|^q ds\right)^{1/q} \leq K, \quad 0 \leq t \tag{5}$$

*for some $K>0$.*

**Nec.** Inequality (3) must hold, but since, in particular ($b=0$), all the solutions of (2) are bounded, then $X_1=R^n$, $X_2=\{0\}$, and (3) becomes (5).

**Suff.** When $q=\infty$, (5) can be written

$$\sup_{0 \leq s \leq t} |Y(t)Y^{-1}(s)| \leq K$$

hence $|Y(t)| \leq K$, i.e. all the solutions of (2) are bounded. We arrive at the same conclusion when $1 \leq q < \infty$, by an adaptation of a Lemma by Coppel ([3], or [4] p. 68). Namely, put

$$\varphi(t) = |Y(t)|^{-1}, \quad \psi(t) = \int_0^t \varphi(s)ds.$$

From the identity

$$\psi(t)Y(t) = \int_0^t Y(t)Y^{-1}(s)Y(s)\varphi(s)ds$$

it follows

$$\psi(t)\varphi^{-1}(t) \leq Kt^{1-1/q} \tag{6}$$

hence

$$d[\psi(t)\exp(-qK^{-1}t^{1/q})]/dt \geq 0$$

and integrating between $\tau>0$ and $t$ we get

$$\psi^{-1}(t) \leq \psi^{-1}(\tau)\exp(qK^{-1}\tau^{1/q})\exp(-qK^{-1}t^{1/q}).$$

Then by (6), or rather by

$$|Y(t)| \leq \psi^{-1}(t)Kt^{1-1/q}$$

it follows

$$|Y(t)| \leq K\psi^{-1}(\tau)\exp(qK^{-1}\tau^{1/q})t^{1-1/q}\exp(-qK^{-1}t^{1/q}).$$

Summing up, both for $q=\infty$ and for $1 \leq q < \infty$, we have $X_1=R^n$, $X_2=\{0\}$, so that (5) coincides with (3) and this, by Theorem 1, means that equation (1) has at least one bounded solution $x$. But then (1) cannot have an unbounded solution $\tilde{x}$ otherwise $\tilde{x}-x$ would be an unbounded solution of equation (2).

4. From Th. 1 also follows

**Theorem 3.** *There is only one bounded solution of equation* (1) *for each* $b \in L^p$, $1 < p \leq \infty$, *if and only if*

$$\left(\int_t^\infty |Y(t)Y^{-1}(s)|^q ds\right)^{1/q} \leq K, \quad 0 \leq t \tag{7}$$

*for some* $K > 0$.

**Nec.** Inequality (3) must hold, but for $b=0$ the only bounded solution of (2) is the zero one, hence $X_1 = \{0\}$, $X_2 = R^n$ and (3) becomes (7).

**Suff.** Putting (Cfr. Coppel [4], p. 74) $\varphi(t) = |Y(t)\xi|^{-q}$, $\xi$ any vector $\neq 0$, We have

$$\left(\int_t^T \varphi(s)ds\right) Y(t)\xi = \int_t^T \varphi(s) Y(t) Y^{-1}(s) Y(s)\xi \, ds$$

hence

$$\varphi^{-1}(t) \int_t^T \varphi(s)ds \leq K^q$$

so that $\int_t^\infty \varphi(s)ds$ exists, which implies $\underline{\lim}\, \varphi(s) = 0$ and consequently $\overline{\lim} |Y(t)\xi| = \infty$ for any vector $\xi \neq 0$. Therefore $X_1 = \{0\}$, $X_2 = R^n$, so that (7) coincides with (3) and equation (1) must have, by Th. 1, at least one bounded solution for each $b \in L^p$. If, for some $b$, there were two different bounded solutions, then their difference would be a bounded, non zero, solution of (2).

**Remark.** Among other helpful criticisms, I owe Professor Coppel the remark that for $q = \infty$ inequality (7), i.e.

$$\sup_{t \leq s} |Y(t)Y^{-1}(s)| \leq K$$

though necessary, is not sufficient to insure that (1) have only one bounded solution for each $b \in L^1$. This is shown by taking $Y(t) = e^{it}$ $(n=1)$.

### References

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[3] W.A. Coppel, Stability and Asymptotic Behavior of Differential Equations, Heath Math. Monographs, Boston (1965).

[4] J.L. Massera, J.J. Schäffer, Annals of Math., **67** (1958), 517-573.

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